MINORS OF NON-COMMUTATIVE SCHEMES

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Abstract. In this article, we develop the theory of minors of non-commutative schemes. This study is motivated by applications in the theory of non-commutative resolutions of singularities of commutative schemes.

Contents

1. Introduction

Let B be a ring and P be a finitely generated projective left B-module. We call the ring $A = A_P = \big(\mathrm{End}_B P\big)^{\mathrm{op}}$ a *minor* of B. It turns out that the module categories of B and A are closely related.

(1) The functors $\mathsf{F} = P \otimes_A \_$ and $\mathsf{H} = \mathrm{Hom}_A(P^\vee, \_)$ from A-Mod to B-Mod are fully faithful, where $P^\vee = \mathrm{Hom}_B(P, B)$. In other words, A-Mod can be realized in two different ways as a full subcategory of B-Mod, see Theorem 4.3.

(2) The functor $\mathsf{G} = \mathrm{Hom}_B(P, \_) : B\text{-}\mathsf{Mod} \longrightarrow A\text{-}\mathsf{Mod}$ is exact and essentially surjective. Moreover, we have adjoint pairs (F, G) and (G, H). In other words, G is a *bilocalization functor*. If

$$I = I_P = \mathrm{Im}\big(P \otimes_A P^\vee \longrightarrow B\big)$$

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and $\bar{B} = B/I$ then the category $\bar{B}$-Mod is the kernel of G and A-Mod is equivalent to the Serre quotient of B-Mod modulo $\bar{B}$-Mod, see Theorem 4.5.

(3) Under certain additional assumptions one can show that the global dimension of B is finite provided the global dimensions of A and $\bar{B}$ are finite, see Lemma 4.9.

The described picture becomes even better when we pass to the (unbounded) derived categories $\mathscr{D}(A\text{-}\mathsf{Mod})$, $\mathscr{D}(B\text{-}\mathsf{Mod})$ and $\mathscr{D}(\bar{B}\text{-}\mathsf{Mod})$ of the rings A, B and $\bar{B}$ introduced above. Let DG be the derived functor of G, LF be the left derived functor of F and RH be the right derived functor of H.

(1) Then we have adjoint pairs $(\mathsf{LF}, \mathsf{DG})$ and $(\mathsf{DG}, \mathsf{RH})$, the functors LF and RH are fully faithful and the category $\mathscr{D}(A\text{-}\mathsf{Mod})$ is equivalent to the Verdier localization of $\mathscr{D}(B\text{-}\mathsf{Mod})$ modulo its triangulated subcategory $\mathscr{D}_{\bar{B}}(B\text{-}\mathsf{Mod})$ consisting of complexes with cohomologies from $\bar{B}$-Mod, see Theorem 4.6.

(2) Moreover, we have a semi-orthogonal decomposition
$$\mathscr{D}(B\text{-}\mathsf{Mod}) = \big\langle \mathscr{D}_{\bar{B}}(B\text{-}\mathsf{Mod}),\, \mathscr{D}(A\text{-}\mathsf{Mod})\big\rangle,$$
see Theorem 2.5.

One motivation to deal with minors comes from the theory of non-commutative crepant resolutions. Let A be a commutative normal Gorenstein domain and F be a reflexive A–module such that the ring
$$B = B_F := \mathrm{End}_A(A \oplus F)^{\mathrm{op}} = \begin{pmatrix} A & F \\ F^\vee & E \end{pmatrix}$$
is maximal Cohen–Macaulay over A and of finite global dimension (here $E = \big(\mathrm{End}_A F\big)^{\mathrm{op}}$). Van den Bergh suggested to view B as a *non-commutative crepant resolution* of A showing that under some additional assumptions, the existence of a non-commutative crepant resolution implies the existence of a commutative one [34]. If we take the idempotent $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in B$ and pose $P = Be$ then it is easy to see that $A = B_P$. Thus, dealing with non-commutative (crepant) resolutions of singularities, we naturally come into the framework of the theory of minors.

In [15] it was observed that there is a close relation between coherent sheaves over the nodal cubic $C = V(zy^2 - x^3 - x^2 z) \subset \mathbb{P}^2$ and representations of the finite dimensional algebra Λ given by the quiver with relations
$$\bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bullet \qquad \beta_1\alpha_1 = \beta_2\alpha_2 = 0.$$

An explanation of this fact was given in [8]. Let $\mathcal{I}$ be the ideal sheaf of the singular point of C and $\mathcal{A} = \mathcal{E}nd_C(\mathcal{O} \oplus \mathcal{I})$. Consider the ringed space $(C, \mathcal{A})$ and the category $\mathcal{A}$-mod of coherent left $\mathcal{A}$–modules on C. The the derived category $\mathscr{D}^b(\mathcal{A}\text{-}\mathsf{mod})$ has a tilting complex, whose (opposite) endomorphism algebra is isomorphic to Λ what implies that the categories $\mathscr{D}^b(\mathcal{A}\text{-}\mathsf{mod})$ and

$\mathscr{D}^b(\Lambda\text{-}\mathsf{mod})$ are equivalent. On the other hand, the triangulated category $\mathsf{Perf}(C)$ of *perfect complexes* on C is equivalent to a full subcategory of $\mathscr{D}^b(\mathcal{A}\text{-}\mathsf{mod})$. In fact, we deal here with a sheaf-theoretic version of the construction of minors: the commutative scheme $(C, \mathcal{O})$ is a minor of the non-commutative scheme $(C, \mathcal{A})$. The goal of this article is to establish a general framework for the theory of minors of non-commutative schemes.

In Section 2 we review some key results on localizations of abelian and triangulated categories used in this article. In Section 3 we discuss the theory of non-commutative schemes, elaborating in particular a proof of the result characterizing the triangulated category $\mathsf{Perf}(\mathcal{A})$ of perfect complexes over a non-commutative scheme $(X, \mathcal{A})$ as the category of *compact objects* of the unbounded derived category of quasi-coherent sheaves $\mathscr{D}(\mathcal{A})$ (Theorem 3.14). Section 4 is devoted to the definition of a minor $(X, \mathcal{A})$ of a non-commutative scheme $(X, \mathcal{B})$ and the study of relations between $(X, \mathcal{A})$ and $(X, \mathcal{B})$. In Section 5 we elaborate the theory of *strongly Gorenstein* non-commutative schemes and the final Section 6 deals with various properties of non-commutative curves.

2. Bilocalizations

Recall that a full subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is said to be *thick* (or *Serre subcategory*) if, for any exact sequence $0 \to C' \to C \to C'' \to 0$, the object C belongs to $\mathscr{C}$ if and only if both C' and C'' belong to $\mathscr{C}$. Then the *quotient category* $\mathscr{A}/\mathscr{C}$ is defined and we denote by $\Pi_{\mathscr{C}}$ the natural functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$. It is exact, essentially surjective and $\operatorname{Ker}\Pi_{\mathscr{C}} = \mathscr{C}$. For instance, if $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ is an exact functor among abelian categories, its kernel $\operatorname{Ker}\mathsf{G}$ is a thick subcategory of $\mathscr{A}$ and G factors as $\bar{\mathsf{G}} \circ \Pi_{\operatorname{Ker}\mathsf{G}}$, where $\bar{\mathsf{G}} : \mathscr{A}/\operatorname{Ker}\mathsf{F} \to \mathsf{B}$.

Analogously, if $\mathscr{C}$ is a full subcategory of a triangulated category $\mathscr{A}$, it is said to be *thick* if it is triangulated (i.e. closed under shifts and cones) and closed under direct summands. Then the *quotient triangulated category* $\mathscr{A}/\mathscr{C}$ is defined and we denote by $\Pi_{\mathscr{C}}$ the natural functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$. It is exact (triangulated), essentially surjective and $\operatorname{Ker}\Pi_{\mathscr{C}} = \mathscr{C}$. For instance, if $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ is an exact (triangulated) functor among triangulated categories, its kernel $\operatorname{Ker}\mathsf{G}$ is a thick subcategory of $\mathscr{A}$ and G factors as $\bar{\mathsf{G}} \circ \Pi_{\operatorname{Ker}\mathsf{G}}$, where $\bar{\mathsf{G}} : \mathscr{A}/\operatorname{Ker}\mathsf{G} \to \mathsf{B}$.

If $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is a functor, we denote by $\operatorname{Im}\mathsf{F}$ its *essential image*, i.e. the full subcategory of $\mathscr{B}$ consisting of objects B such that there is an isomorphism $B \simeq \mathsf{F}A$ for some $A \in \mathscr{A}$. We usually use this term when F is a full embedding (i.e. is fully faithful), so $\operatorname{Im}\mathsf{F} \simeq \mathscr{A}$.

We use the following well-known facts related to these notions.

Theorem 2.1. (1) *Let $\mathscr{A}, \mathscr{B}$ be abelian categories, $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ be an exact functor and $\mathsf{F} : \mathscr{B} \to \mathscr{A}$ be its left adjoint (right adjoint) such that the natural morphism $\mathbb{1}_{\mathscr{B}} \to \mathsf{G} \circ \mathsf{F}$ (respectively, $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{B}}$) is an isomorphism. Let $\mathscr{C} = \operatorname{Ker}\mathsf{G}$.*

(a) $\mathsf{G} = \bar{\mathsf{G}} \circ \Pi_{\mathscr{C}}$, *where* $\bar{\mathsf{G}}$ *is an equivalence* $\mathscr{A}/\mathscr{C} \xrightarrow{\sim} \mathscr{B}$ *and its quasi-inverse functor is* $\bar{\mathsf{F}} = \Pi_{\mathscr{C}} \circ \mathsf{F}$.

(b) F *is a full embedding and its essential image* $\operatorname{Im}\mathsf{F}$ *coincides with the left (respectively, right) orthogonal subcategory of* $\mathscr{C}$, *i.e. the full subcategory*

$$^{\perp}\mathscr{C} = \{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(A,C) = \operatorname{Ext}^1(A,C) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C}\,\}$$

(*respectively,*

$$\mathscr{C}^{\perp} = \{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(C,A) = \operatorname{Ext}^1(C,A) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C}\,\}.)$$

(c) $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$ (*respectively,* $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$).

(d) *The embedding functor* $\mathscr{C} \to \mathscr{A}$ *has a left (respectively, right) adjoint.*

(2) *Let* $\mathscr{A}, \mathscr{B}$ *be triangulated categories,* $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ *be an exact (triangulated) functor and* $\mathsf{F} : \mathscr{B} \to \mathscr{A}$ *be its left adjoint (right adjoint) such that the natural morphism* $\mathbb{1}_{\mathscr{B}} \to \mathsf{G} \circ \mathsf{F}$ (*respectively,* $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{B}}$) *is an isomorphism. Let* $\mathscr{C} = \operatorname{Ker}\mathsf{G}$.

(a) $\mathsf{G} = \bar{\mathsf{G}} \circ \Pi_{\mathscr{C}}$, *where* $\bar{\mathsf{G}}$ *is an equivalence* $\mathscr{A}/\mathscr{C} \xrightarrow{\sim} \mathscr{B}$ *and its quasi-inverse functor is* $\bar{\mathsf{F}} = \Pi_{\mathscr{C}} \circ \mathsf{F}$.

(b) F *is a full embedding and its essential image* $\operatorname{Im}\mathsf{F}$ *coincides with the left (respectively, right) orthogonal subcategory of* $\mathscr{C}$, *i.e. the full subcategory*[1]

$$^{\perp}\mathscr{C} = \{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(A,C) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C}\,\}$$

(*respectively,*

$$\mathscr{C}^{\perp} = \{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(C,A) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C}\,\}.)$$

(c) $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$ (*respectively,* $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$).

(d) *The embedding functor* $\mathscr{C} \to \mathscr{A}$ *has a left (respectively, right) adjoint, which induces an equivalence* $\mathscr{A}/{}^{\perp}\mathscr{C} \xrightarrow{\sim} \mathscr{C}$ (*respectively,* $\mathscr{A}/\mathscr{C}^{\perp} \xrightarrow{\sim} \mathscr{C}$).

Proof. The statement (1a) is proved in [16, Ch. III, Proposition 5] if F is right adjoint of G. The case of left adjoint is just a dualization. The proof of the statement (2a) is quite analogous. Therefore, from now on we can suppose that $\mathscr{B} = \mathscr{A}/\mathscr{C}$. Then the statements (1b) and (2b) are just [16, Ch. III, Lemma 2 et Corollaire] and [29, Theorem 9.1.16]. The statements (1c) and (2c) are [18, Corollary 2.3] and [29, Corollary 9.1.14]. Thus the statement (2d) also follows from [29, Theorem 9.1.16]. In the abelian case the left (respectively, right) adjoint J to the embedding $\mathscr{C} \to \mathscr{A}$ is given by the rule $A \mapsto \operatorname{Cok}\Psi(A)$ (respectively, $A \mapsto \operatorname{Ker}\Psi(A)$), where Ψ is the natural morphism $\mathsf{F} \circ \mathsf{G} \to \mathbb{1}_{\mathscr{A}}$ (respectively, $\mathbb{1}_{\mathscr{A}} \to \mathsf{F} \circ \mathsf{G}$). $\square$

[1] Note that in the book [29] the notations for the orthogonal subcategories are opposite to ours. The latter seems more usual, especially in the representation theory, see, for instance, [2, 18]. In [16] the objects of the right orthogonal subcategory $\mathscr{C}^{\perp}$ are called *$\mathscr{C}$-closed*.

Remark. Note that in the abelian case the composition $\Pi_{{}^{\perp}\mathscr{C}} \circ \mathsf{J}$ (respectively, $\Pi_{\mathscr{C}^{\perp}} \circ \mathsf{J}$) need not be an equivalence. The reason is that the subcategory ${}^{\perp}\mathscr{C}$ ($\mathscr{C}^{\perp}$) need not be thick (see [18]).

A thick subcategory $\mathscr{C}$ of an abelian or triangulated category $\mathscr{A}$ is said to be *localizing* (*colocalizing*) if the canonical functor $\mathsf{G} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ has a right (respectively, left) adjoint F. Neeman [29] calls F a *Bousfield localization* (respectively, a *Bousfield colocalization*).[2] In this case the natural morphism $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{A}/\mathscr{C}}$ (respectively, $\mathbb{1}_{\mathscr{A}/\mathscr{C}} \to \mathsf{G} \circ \mathsf{F}$) is an isomorphism [16, Ch.III,Proposition 3], [29, Lemma 9.1.7]. If $\mathscr{C}$ is both localizing and colocalizing, we call it *bilocalizing* and call the category $\mathscr{A}/\mathscr{C}$ (or any equivalent one) a *bilocalization* of $\mathscr{A}$. We also say in this case that G is a *bilocalization functor*. In other words, an exact functor $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ is a bilocalization functor if it has both left adjoint F and right adjoint H and the natural morphisms $\mathbb{1}_{\mathscr{B}} \to \mathsf{GF}$ and $\mathsf{GH} \to \mathbb{1}_{\mathscr{B}}$ are isomorphisms.

Corollary 2.2. *Let $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ be an exact functor between abelian or triangulated categories which has both left adjoint F and right adjoint H. In order that G will be a bilocalization functor it is necessary and sufficient that one of the natural morphisms $\mathbb{1}_{\mathscr{B}} \to \mathsf{GF}$ or $\mathsf{GH} \to \mathbb{1}_{\mathscr{B}}$ be an isomorphism.*

Proof. Let, for instance, the first of these morphisms be an isomorphism. Then there is an equivalence of categories $\bar{\mathsf{G}} : \mathscr{A}/\operatorname{Ker}\mathsf{G} \xrightarrow{\sim} \mathsf{B}$ such that $\mathsf{G} = \bar{\mathsf{G}}\Pi_{\mathscr{C}}$, where $\mathscr{C} = \operatorname{Ker}\mathsf{G}$. So we can suppose that $\mathscr{B} = \mathscr{A}/\mathscr{C}$ and $\mathsf{G} = \Pi_{\mathscr{C}}$. Thus the morphism $\mathsf{GH} \to \mathbb{1}_{\mathscr{B}}$ is an isomorphism, since H is right adjoint to G. □

Corollary 2.3. *Let $\mathscr{C}$ be a localizing (colocalizing) thick subcategory of an abelian category $\mathscr{A}$, $\mathscr{D}_{\mathscr{C}}(\mathscr{A})$ be the full subcategory of $\mathscr{D}(\mathscr{A})$ consisting of all complexes $C^{\bullet}$ such that all cohomologies $H^i(C^{\bullet})$ are in $\mathscr{C}$. Suppose that the Bousfield localization (respectively, colocalization) functor F has right (respectively, left) derived functor. Then $\mathscr{D}_{\mathscr{C}}(\mathscr{A})$ is also a localizing (colocalizing) subcategory of $\mathscr{A}$ and $\mathscr{D}(\mathscr{A})/\mathscr{D}_{\mathscr{C}}(\mathscr{A}) \simeq \mathscr{D}(\mathscr{A}/\mathscr{C})$.*

Proof. We consider the case of a localizing subcategory $\mathscr{C}$, denote by G the canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ and by F its right adjoint. As G is exact, it induces an exact functor $\mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{A}/\mathscr{C})$ acting on complexes componentwise. We denote it by DG; it is both right and left derived of G. Obviously, $\operatorname{Ker}\mathsf{DG} = \mathscr{D}_{\mathscr{C}}(\mathscr{A})$. Since $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{A}/\mathscr{C}}$ is an isomorphism, the morphism $\mathsf{DG} \circ \mathsf{RF} \to \mathbb{1}_{\mathscr{D}(\mathscr{A}/\mathscr{C})}$ is also an isomorphism, so we can apply Theorem2.1 (2). □

Remark 2.4. (1) If $\mathscr{C}$ is localizing and $\mathscr{A}$ is a Grothendieck category, the right derived functor RF exists [2], so $\mathscr{D}(\mathscr{A}/\mathscr{C}) \simeq \mathscr{D}_{\mathscr{C}}(\mathscr{A})$. We do not know general conditions which ensure the existence of the left derived functor LF in the case of colocalizing categories, though it

[2] Actually, Neeman uses this term for triangulated categories, but we will use it for abelian categories too.

exists when $\mathscr{A}$ is a category of quasi-coherent modules over a quasi-compact separated non-commutative scheme and F is tensor product or inverse image, see Proposition 3.12.

(2) Miyatchi [27] proved that always $\mathscr{D}^\sigma(\mathscr{A}/\mathscr{C}) \simeq \mathscr{D}^\sigma_{\mathscr{C}}(\mathscr{A})$, where $\sigma \in \{+,-,b\}$.

We recall that a sequence $(\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_m)$ of triangulated subcategories of a triangulated category $\mathscr{A}$ is said to be a *semi-orthogonal decomposition* of $\mathscr{A}$ if $\mathscr{A}(A, A') = 0$ for $A \in \mathscr{A}_i$, $A' \in \mathscr{A}_j$ and $i > j$, and for every object $A \in \mathscr{A}$ there is a chain of morphisms

$$0 = A_m \xrightarrow{f_m} A_{m-1} \xrightarrow{f_{m-1}} \dots A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 = A$$

such that $\operatorname{Cone} f_i \in \mathscr{A}_i$ [24].

Theorem 2.5. *Let $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ be an exact functor among triangulated categories, $\mathsf{F} : \mathscr{B} \to \mathscr{A}$ be its right* (*left*) *adjoint such that the natural morphism $\phi : \mathbb{1}_{\mathscr{B}} \to \mathsf{GF}$* (*respectively, $\psi : \mathsf{GF} \to \mathbb{1}_{\mathscr{B}}$*) *is an isomorphism. Then* $(\operatorname{Im} \mathsf{F}, \operatorname{Ker} \mathsf{G})$ (*respectively,* $(\operatorname{Ker} \mathsf{G}, \operatorname{Im} \mathsf{F})$) *is a semi-orthogonal decomposition of $\mathscr{A}$.*

Proof. We consider the case of left adjoint. If $A = \mathsf{F}B$ and $A' \in \operatorname{Ker} \mathsf{G}$, then $\mathscr{A}(A, A') \simeq \mathscr{B}(\mathsf{G}A, B) = 0$. On the other hand, consider the natural morphism $f : \mathsf{FG}A \to A$. Then $\mathsf{G}f$ is an isomorhism, whence $\operatorname{Cone} f \in \operatorname{Ker} \mathsf{G}$. So we can set $A_1 = \mathsf{FG}A$, $f_1 = f$. □

3. Non-commutative schemes

Definition 3.1. (1) A *non-commutative scheme* is a pair $(X, \mathcal{A})$, where X is a scheme (called the *commutative background* of the non-commutative scheme) and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, which is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. Sometimes we say "non-commutative scheme $\mathcal{A}$" not mentioning its commutative background X. We denote by X_{cl} the set of closed points of X.

(2) A non-commutative scheme $(X, \mathcal{A})$ is said to be *affine* (*separated, quasi-compact*) if so is its commutative background X. It is said to be *reduced* if $\mathcal{A}$ has no nilpotent ideals.

(3) A *morphism* of non-commutative schemes $f : (Y, \mathcal{B}) \to (X, \mathcal{A})$ is a pair $(f_X, f^\#)$, where $f_X : Y \to X$ is a morphism of schemes and $f^\#$ is a morphism of $f_X^{-1}\mathcal{O}_X$-algebras $f_X^{-1}\mathcal{A} \to \mathcal{B}$. In what follows we usually write f instead of f_X.

(4) Given a non-commutative scheme $(X, \mathcal{A})$, we denote by $\mathcal{A}\text{-}\mathsf{Mod}$ (respectively, by $\mathcal{A}\text{-}\mathsf{mod}$) the category of quasi-coherent (respectively, coherent) sheaves of $\mathcal{A}$-modules. We call objects of this category just $\mathcal{A}$-modules (respectively, coherent $\mathcal{A}$-modules).

(5) If $f : (Y, \mathcal{B}) \to (X, \mathcal{A})$ is a morphism of non-commutative schemes, we denote by $f^* : \mathcal{A}\text{-}\mathsf{Mod} \to \mathcal{B}\text{-}\mathsf{Mod}$ the functor of inverse image which maps an $\mathcal{A}$-module $\mathcal{M}$ to the $\mathcal{B}$-module $\mathcal{B} \otimes_{f^{-1}\mathcal{A}} f^{-1}\mathcal{M}$. If the

map f_X is separated and quasi-compact, we denote by $f_* : \mathcal{B}\text{-}\mathsf{Mod} \to \mathcal{A}\text{-}\mathsf{Mod}$ the functor of direct image. It follows from [19, § 0.1 and § 1.9.2] that these functors are well-defined. Moreover, f^* maps coherent modules to coherent ones.

In this paper we always suppose non-commutative schemes *separated and quasi-compact*.

Remark 3.2. (1) If $(X, \mathcal{A})$ is affine, i.e. $X = \operatorname{Spec} \boldsymbol{R}$ for some commutative ring $\boldsymbol{R}$, then $\mathcal{A} = \boldsymbol{A}^\sim$ is a sheafification of an $\boldsymbol{R}$-algebra $\boldsymbol{A}$. A quasi-coherent $\mathcal{A}$-module is just a sheafification $M^\sim$ of an $\boldsymbol{A}$-module M, so $\mathcal{A}\text{-}\mathsf{Mod} \simeq \boldsymbol{A}\text{-}\mathsf{Mod}$ and we identify these categories. If, moreover, $\boldsymbol{A}$ is noetherian, then $\mathcal{A}\text{-}\mathsf{mod}$ coincides with the category $\boldsymbol{A}\text{-}\mathsf{mod}$ of finitely generated $\boldsymbol{A}$-modules.

(2) If X is separated and quasi-compact, $\mathcal{A}\text{-}\mathsf{Mod}$ is a *Grothendieck category*. In particular, every quasi-coherent $\mathcal{A}$-module has an injective envelope. We denote by $\mathcal{A}\text{-}\mathsf{Inj}$ the full subcategory of $\mathcal{A}\text{-}\mathsf{Mod}$ consisting of injective modules.

(3) The inverse image functor f^* for a morphism of non-commutative schemes usually does not coincide with the inverse image functor f_X^* with respect to the morphism of their commutative backgrounds. We can guarantee it if $\mathcal{B} = f_X^* \mathcal{A}$, for instance, if Y is an open subset of X and $\mathcal{B} = \mathcal{A}|_Y$.

Definition 3.3. (1) The *center* of $\mathcal{A}$ is the subsheaf $\operatorname{cen} \mathcal{A} \subseteq \mathcal{A}$ such that $(\operatorname{cen} \mathcal{A})(U) = \{\, \alpha \in \mathcal{A}(U) \mid \alpha|V \in \operatorname{cen} \mathcal{A}(V) \text{ for all } V \subseteq U \,\}$, where $\operatorname{cen} \boldsymbol{A}$ denotes the center of a ring $\boldsymbol{A}$.

(2) We say that a non-commutative scheme $(X, \mathcal{A})$ is *central*, if the natural homomorphism $\mathcal{O}_X \to \mathcal{A}$ maps $\mathcal{O}_X$ bijectively onto the center $\operatorname{cen}(\mathcal{A})$ of $\mathcal{A}$.

Note that if $(X, \mathcal{A})$ is affine, $X = \operatorname{Spec} \boldsymbol{R}$ and $\mathcal{A} = \boldsymbol{A}^\sim$, then $\operatorname{cen} \mathcal{A} = (\operatorname{cen} \boldsymbol{A})^\sim$.

Proposition 3.4. $\operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}} \simeq \operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Inj}} \simeq \Gamma(X, \operatorname{cen} \mathcal{A})$.

Proof. Let $\alpha \in \Gamma(X, \operatorname{cen} \mathcal{A})$. Given any $\mathcal{M} \in \mathcal{A}\text{-}\mathsf{Mod}$, define $\alpha(\mathcal{M}) : \mathcal{M} \to \mathcal{M}$ by the rule: $\alpha(M)(U) : \mathcal{M}(U) \to \mathcal{M}(U)$ is the multiplication by $\alpha|U$ for every open $U \subseteq X$. Obviously, it is a morphism of $\mathcal{A}$-modules. Moreover, if $f \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$, one easily sees that $f\alpha(M) = \alpha(N)f$, so α defines an element from $\operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}}$.

Conversely, let $\lambda \in \operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}}$. Let $U \subseteq X$ be an open subset, $j : U \to X$ be the embedding. Then $\lambda(U) = \lambda(j_* j^* \mathcal{A})$ is an element from $\operatorname{End}_{\mathcal{A}}(j_* j^* \mathcal{A}) = \mathcal{A}(U)$. Evidently, it is in $\operatorname{cen} \mathcal{A}(U)$. Moreover, if $V \subseteq U$ is another open subset, $j' : V \to X$ is the embedding, the restriction

homomorphism $r : j_* j^* \mathcal{A} \to j'_* j'^* \mathcal{A}$ gives the commutative diagram

$$\begin{tikzcd} j_* j^* \mathcal{A} \arrow[r, "\lambda(U)"] \arrow[d, "r"] & j_* j^* \mathcal{A} \arrow[d, "r"] \\ j'_* j'^* \mathcal{A} \arrow[r, "\lambda(V)"] & j'_* j'^* \mathcal{A} \end{tikzcd}$$

It implies that $\lambda(V) = \lambda(U)|V$. In particular, $\lambda(X) = \alpha$ is an element from $\Gamma(X, \operatorname{cen} \mathcal{A})$ and $\lambda(U)$ coincides with the multiplication by $\alpha|U$. Thus we obtain an isomorphism $\operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}} \simeq \Gamma(X, \operatorname{cen} \mathcal{A})$.

There is the restriction map $\operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}} \to \operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Inj}}$. On the other hand, consider an injective copresentation of an $\mathcal{A}$-module $\mathcal{M}$, i.e. an exact sequence $0 \to \mathcal{M} \xrightarrow{\alpha_{\mathcal{M}}} \mathcal{I}_{\mathcal{M}} \to \mathcal{I}'_{\mathcal{M}}$ with injective modules $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}'_{\mathcal{M}}$. Let $\lambda \in \operatorname{End} \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Inj}}$. Then there is a unique homomorphism $\lambda(\mathcal{M}) : \mathcal{M} \to \mathcal{M}$ such that $\lambda(\mathcal{I}_{\mathcal{M}})\alpha_{\mathcal{M}} = \alpha_{\mathcal{M}}\lambda(\mathcal{M})$. Let $0 \to \mathcal{N} \xrightarrow{\alpha_{\mathcal{N}}} \mathcal{I}_{\mathcal{N}} \to \mathcal{I}'_{\mathcal{N}}$ be an injective copresentation of another $\mathcal{A}$-module $\mathcal{N}$ and $f \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$. Extending f to injective copresentations, we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r, "\alpha_{\mathcal{M}}"] \arrow[d, "f"] & \mathcal{I}_{\mathcal{M}} \arrow[r] \arrow[d, "f_0"] & \mathcal{I}'_{\mathcal{M}} \arrow[d, "f_1"'] \\ 0 \arrow[r] & \mathcal{N} \arrow[r, "\alpha_{\mathcal{N}}"] & \mathcal{I}_{\mathcal{N}} \arrow[r] & \mathcal{I}'_{\mathcal{N}} \end{tikzcd}$$

It implies that

$$\begin{aligned} \alpha_{\mathcal{N}}\lambda(\mathcal{N})f = \lambda(\mathcal{I}_{\mathcal{N}})\alpha_{\mathcal{N}}f = \lambda(\mathcal{I}_{\mathcal{N}})f_0\alpha_{\mathcal{M}} = \\ = f_0\lambda(\mathcal{I}_{\mathcal{M}})\alpha_{\mathcal{M}} = f_0\alpha_{\mathcal{M}}\lambda(\mathcal{M}) = \alpha_{\mathcal{N}}f\lambda(\mathcal{M}), \end{aligned}$$

whence $\lambda(\mathcal{N})f = f\lambda(\mathcal{M})$, so we have extended λ to a unique endomorphism of $\mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}}$. □

Proposition 3.5. *Let $\mathcal{C} = \operatorname{cen}(\mathcal{A})$, $X' = \operatorname{Spec} \mathcal{C}$ be the spectrum of the (commutative) $\mathcal{O}_X$-algebra $\mathcal{C}$, $\phi : X' \to X$ be the structural morphism, and $\mathcal{A}' = \phi^{-1}\mathcal{A}$.*

1. *$\mathcal{A}'$ is an $\mathcal{O}_{X'}$-algebra, so $(X', \mathcal{A}')$ is a central non-commutative scheme.*
2. *For any $\mathcal{F} \in \mathcal{A}$-Mod the natural map $\mathcal{F} \to \phi_*\phi^*\mathcal{F}$ is an isomorphism.*[3]
3. *For any $\mathcal{F}' \in \mathcal{A}'$-Mod the natural map $\phi^*\phi_*\mathcal{F}' \to \mathcal{F}'$ is an isomorphism.*
4. *The functors ϕ^* and ϕ_* establish an equivalence of the categories $\mathcal{A}$-Mod and $\mathcal{A}$-Mod$'$ as well as of $\mathcal{A}$-mod and $\mathcal{A}'$-mod.*

Thus, when necessary, we can suppose, without loss of generality, that our non-commutative schemes are central.

[3] Note that in this situation $\phi^* = \phi^{-1}$.

Proof. All claims are obviously local, so we can suppose that $X = \operatorname{Spec} \boldsymbol{R}$ and $X' = \operatorname{Spec} \boldsymbol{R}'$, where $\boldsymbol{R}'$ is the center of the $\boldsymbol{R}$-algebra $\boldsymbol{A} = \Gamma(X, \mathcal{A})$. Then all claims are trivial. $\square$

We call a non-commutative scheme $(X, \mathcal{A})$ *noetherian* if the scheme X is noetherian and $\mathcal{A}$ is coherent as a sheaf of $\mathcal{O}_X$-modules. Note that if $(X, \mathcal{A})$ is noetherian, the central non-commutative scheme $(X', \mathcal{A}')$ constructed in Proposition 3.5 is also noetherian. In particular, if an affine non-commutative scheme $(\operatorname{Spec} \boldsymbol{R}, \boldsymbol{A}^{\sim})$ is noetherian, then $\boldsymbol{A}$ is a *noetherian algebra*, i.e. $\boldsymbol{C} = \operatorname{cen} \boldsymbol{A}$ is noetherian and $\boldsymbol{A}$ is a finitely generated $\boldsymbol{C}$-module.

Definition 3.6. Let $(X, \mathcal{A})$ be noetherian.

(1) We denote by $\mathsf{lp}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of *locally projective* modules $\mathcal{P}$, i.e. such that $\mathcal{P}_x$ is a projective $\mathcal{A}_x$-module for every $x \in X$.

(2) We say that $\mathcal{A}$ *has enough locally projective modules* if for every coherent $\mathcal{A}$-module $\mathcal{M}$ there is an epimorphism $\mathcal{P} \to \mathcal{M}$, where $\mathcal{P} \in \mathsf{lp}\,\mathcal{A}$. Since every quasi-coherent module is a sum of its coherent submodules, then for every quasi-coherent $\mathcal{A}$-module $\mathcal{M}$ there is an epmorphism $\mathcal{P} \to \mathcal{M}$, where $\mathcal{P}$ is a coproduct of modules from $\mathsf{lp}\,\mathcal{A}$.

An important example arises as follows. We say that a noetherian non-commutative scheme $(X, \mathcal{A})$ is *quasi-projective* if there is an ample $\mathcal{O}_X$-module $\mathcal{L}$ [20, Section 4.5]. Note that in this case X is indeed a quasi-projective scheme over the ring $R = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes n})$.

Proposition 3.7. *Every quasi-projective non-commutative scheme* $(X, \mathcal{A})$ *has enough locally projective modules.*

Proof. Let $\mathcal{L}$ be an ample $\mathcal{O}_X$-module, $\mathcal{M}$ be any coherent $\mathcal{A}$-module. There is an epimorphism of $\mathcal{O}_X$-modules $n\mathcal{O}_X \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$ for some m, hence also an epimorphism $\mathcal{F} = n\mathcal{L}^{\otimes(-m)} \to \mathcal{M}$. Since $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M})$, it gives an epimorphism of $\mathcal{A}$-modules $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{M}$, where $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \in \mathsf{lp}\,\mathcal{A}$. $\square$

We define an *invertible $\mathcal{A}$-module* as an $\mathcal{A}$-module $\mathcal{I}$ such that $\operatorname{End}_{\mathcal{A}} \mathcal{I} \simeq \mathcal{A}^{\mathrm{op}}$ and the natural map $\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{I} \to (\operatorname{End}_{\mathcal{A}} \mathcal{I})^{\mathrm{op}} \simeq \mathcal{A}$ is an isomorphism. For instance, the modules constructed in the preceding proof are invertible. On the contrary, one easily proves that, if $\mathcal{A}$ is noetherian and $\operatorname{cen} \mathcal{A} = \mathcal{O}_X$, any invertible $\mathcal{A}$-module $\mathcal{I}$ is isomorphic to $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{L} = \mathcal{H}om_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{I}, \mathcal{I})$ and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. (We will not use this fact.)

We denote by $\mathscr{C}\mathcal{A}$ the category of complexes of $\mathcal{A}$-modules, by $\mathscr{K}\mathcal{A}$ the category of complexes modulo homotopy and by $\mathscr{D}\mathcal{A}$ the derived category $\mathscr{D}(\mathcal{A}\text{-}\mathsf{Mod})$. We also use the conventional notations $\mathscr{C}^{\sigma}\mathcal{A}$ $\mathscr{K}^{\sigma}\mathcal{A}$ and $\mathscr{D}^{\sigma}\mathcal{A}$, where $\sigma \in \{+, -, b\}$. We denote by $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ the full subcategory of *compact* objects $\mathcal{C}^{\bullet}$ from $\mathscr{D}\mathcal{A}$, i.e. such that the natural morphism

$$\textstyle\coprod_i \operatorname{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{C}^{\bullet}, \mathcal{F}_i^{\bullet}) \to \operatorname{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{C}^{\bullet}, \textstyle\coprod_i \mathcal{F}_i^{\bullet})$$

is bijective for any coproduct $\coprod_i \mathcal{F}_i^\bullet$.

Recall that a complex $\mathcal{I}^\bullet$ is said to be *K-injective* [33] if for every acyclic complex $\mathcal{C}^\bullet$ the complex $\mathrm{Hom}^\bullet(\mathcal{C}^\bullet, \mathcal{I}^\bullet)$ is acyclic too. We denote by $\mathsf{K\text{-}inj}\,\mathcal{A}$ the full subcategory of $\mathscr{K}\mathcal{A}$ consisting of K-injective complexes and by $\mathsf{K\text{-}inj}_0\,\mathcal{A}$ its full subcategory consisting of acyclic K-injective complexes.

Proposition 3.8. *Let $(X, \mathcal{A})$ be a non-commutative scheme (separated and quasi-compact).*

(1) *For every complex $\mathcal{C}^\bullet$ in $\mathscr{C}\mathcal{A}$ there is a* K-injective resolution*, i.e. a K-injective complex $\mathcal{I}^\bullet \in \mathscr{C}\mathcal{A}$ together with a quasi-isomorphism $\mathcal{C}^\bullet \to \mathcal{I}^\bullet$.*
(2) $\mathscr{D}\mathcal{A} \simeq \mathsf{K\text{-}inj}\,\mathcal{A} / \mathsf{K\text{-}inj}_0\,\mathcal{A}$.

Proof. As the category $\mathcal{A}$-Mod is a Grothendieck category, (1) follows immediately from [2, Theorem 5.4] (see also [33, Lemma 3.7 and Proposition 3.13]). Then (2) follows from [33, Proposition 1.5]. □

A complex $\mathcal{F}^\bullet$ is said to be *K-flat* [33] if for every acyclic complex $\mathcal{S}^\bullet$ of *right* $\mathcal{A}$-modules the complex $\mathcal{F}^\bullet \otimes_{\mathcal{A}} \mathcal{S}^\bullet$ is acyclic. The next result is quite analogous to [1, Proposition 1.1] and the proof just repeats that of the cited paper with no changes.

Proposition 3.9. *Let $(X, \mathcal{A})$ be a non-commutative scheme. Then for every complex $\mathcal{C}^\bullet$ in $\mathscr{C}\mathcal{A}$ there is a* K-flat replica*, i.e. a K-flat complex $\mathcal{F}^\bullet$ quasi-isomorphic to $\mathcal{C}^\bullet$.*

Remark 3.10. If $(X, \mathcal{A})$ is noetherian and has enough locally projective modules, every complex from $\mathscr{C}^-\mathcal{A}$ has a locally projective (hence flat) resolution. Then [33, Theorem 3.4] implies that for every complex $\mathcal{C}$ from $\mathscr{C}\mathcal{A}$ there is an *Lp-resolution*, i.e. a K-flat complex $\mathcal{F}^\bullet$ consisting of locally projective modules together with a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{C}^\bullet$. For instance, it is the case if $(X, \mathcal{A})$ is *quasi-projective* (Proposition 3.7).

A complex $\mathcal{I}^\bullet$ is said to be *weakly K-injective* if for every acyclic K-flat complex $\mathcal{F}^\bullet$ the complex $\mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is exact.

Proposition 3.11 ([33, Propositions 5.4 and 5.15])**.** *Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism of non-commutative scheme.*

(1) *If $\mathcal{F}^\bullet \in \mathscr{C}\mathcal{B}$ is K-flat, then so is also $f^*\mathcal{F}^\bullet$. If, moreover, $\mathcal{F}^\bullet$ is K-flat and acyclic, then $f^*\mathcal{F}^\bullet$ is acyclic too.*
(2) *If $\mathcal{I} \in \mathscr{C}\mathcal{A}$ is weakly K-injective, then $f_*\mathcal{I}$ is weakly K-injective. If, moreover, $\mathcal{I}$ is weakly K-injective and acyclic, then $f_*\mathcal{I}$ is acyclic too.*

Proposition 3.12 (cf. [33, Section 6])**.** *Let $(X, \mathcal{A})$ be a non-commutative scheme.*

(1) *The derived functors $\mathrm{RHom}^\bullet_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and $\mathrm{R}\mathcal{H}om^\bullet_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ exist and can be calculated using a K-injective resolution of $\mathcal{G}^\bullet$ or a weakly K-injective resolution of $\mathcal{G}^\bullet$ and a K-flat replica of $\mathcal{F}^\bullet$.*

(2) *The derived functor* $\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet$, *where* $\mathcal{G}^\bullet \in \mathscr{D}\mathcal{A}^{\mathrm{op}}$, *exists and can be calculated using a K-flat replica either of* $\mathcal{F}$ *or of* $\mathcal{G}$. *Moreover, if* $\mathcal{G}^{\mathrm{op}}$ *is a complex of* $\mathcal{A}$-$\mathcal{B}$*-bimodules, where* $\mathcal{B}$ *is another sheaf of* $\mathcal{O}_X$*-algebras, there are isomorphisms of functors*

$$\mathrm{RHom}_{\mathcal{B}}(\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet, \mathcal{M}^\bullet) \simeq \mathrm{RHom}_{\mathcal{A}}(\mathcal{F}, \mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}^\bullet, \mathcal{M}^\bullet))$$

$$\mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet, \mathcal{M}^\bullet) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}^\bullet, \mathcal{M}^\bullet)).$$

(3) *For every morphism* $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ *the derived functors* $\mathrm{L}f^* : \mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ *and* $\mathrm{R}f_* : \mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$ *exist. They can be calculated using, respectively, K-flat replicas in* $\mathscr{C}\mathcal{B}$ *and weakly K-injective resolutions in* $\mathscr{C}\mathcal{A}$. *Moreover, there are isomorphisms of functors*

$$\mathrm{RHom}^\bullet_{\mathcal{B}}(\mathcal{F}^\bullet, \mathrm{R}f_*\mathcal{G}^\bullet) \simeq \mathrm{RHom}^\bullet_{\mathcal{A}}(\mathrm{L}f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$
$$\mathrm{R}\mathcal{H}om^\bullet_{\mathcal{B}}(\mathcal{F}^\bullet, \mathrm{R}f_*\mathcal{G}^\bullet) \simeq \mathrm{R}f_*\,\mathrm{R}\mathcal{H}om^\bullet_{\mathcal{A}}(\mathrm{L}f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

(4) *If* $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ *is another morphism of non-commutative schemes, then* $\mathrm{L}(g \circ f)^* \simeq \mathrm{L}f^* \circ \mathrm{L}g^*$ *and* $\mathrm{R}(g \circ f)_* \simeq \mathrm{R}g_* \circ \mathrm{R}f_*$.

If the considered non-commutative schemes have enough locally projective modules (*for instance, are quasi-projective*), *one can replace in these statements K-flat replicas by Lp-resolutions.*

In particular, let $f : \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism of rings. We consider $\boldsymbol{B}$ as an algebra over a subring $\boldsymbol{S}$ (an arbitrary one) of its center and $\boldsymbol{A}$ as an algebra over a subring $\boldsymbol{R} \subseteq \mathrm{cen}\,\boldsymbol{A} \cap f^{-1}(\boldsymbol{S})$. Then we can identify f with its sheafification $f^\sim : (\mathrm{Spec}\,\boldsymbol{S}, \boldsymbol{B}^\sim) \to (\mathrm{Spec}\,\boldsymbol{R}, \boldsymbol{A}^\sim)$. In this context the functors $(f^\sim)^*$ and $(f^\sim)_*$ are just sheafifications, respectively, of the “back-up” functor ${}_BM \mapsto {}_AM$ and the “change-of-scalars” functor ${}_AN \mapsto {}_BB \otimes_A N$.

Definition 3.13. A complex $\mathcal{C}^\bullet$ in $\mathscr{C}\mathcal{A}$ is said to be *perfect* if for every point $x \in X$ there is an open neighbourhood U of x such that $\mathcal{C}|_U$ is quasi-isomorphic to a finite complex of locally projective coherent modules. We denote by $\mathsf{Perf}\mathcal{A}$ the full subcategory of $\mathscr{D}\mathcal{A}$ consisting of perfect complexes.

The following result is well-known in commutative and affine cases [28, 32]. Though the proof in non-commutative situation is almost the same, we include it for the sake of completeness. Actually, we reproduce the proof of Rouquier with slight changes.

Theorem 3.14. *Let* $(X, \mathcal{A})$ *be a non-commutative scheme* (*quasi-compact and separated*). *Then* $\mathscr{D}\mathcal{A}$ *is compactly generated and* $\mathscr{D}^{\mathrm{c}}\mathcal{A} = \mathsf{Perf}\mathcal{A}$.

Proof. Let $U \subseteq X$ be an open affine subset of X, $\mathcal{A}_U$ be the restriction of $\mathcal{A}$ onto U, $\complement U = X \setminus U$, $j = j_U : U \to X$ be the embedding. Then the inverse image functor $j^* : \mathcal{A}\text{-}\mathsf{Mod} \to \mathcal{A}_U\text{-}\mathsf{Mod}$ is exact and the natural morphism $j^*j_* \to \mathbb{1}_{\mathcal{A}_U\text{-}\mathsf{Mod}}$ is an isomorphism (actually, identity). Therefore $\mathrm{Ker}\, j^*$ is a localizing subcategory and $\mathcal{A}\text{-}\mathsf{Mod}/\mathrm{Ker}\, j^* \simeq \mathcal{A}_U\text{-}\mathsf{Mod}$. Note that $\mathrm{Ker}\, j_U^*$ consists of the $\mathcal{A}$-modules $\mathcal{M}$ such that $\mathrm{supp}\,\mathcal{M} \subseteq \complement U$. Then $\mathrm{Ker}\,\mathrm{L}j_*$

is a localizing subcategory of $\mathscr{D}\mathcal{A}$ and $\mathscr{D}\mathcal{A}/\operatorname{Ker}\mathrm{L}j_* \simeq \mathscr{D}\mathcal{A}_U$. This kernel coincide with the full subcategory $\mathscr{D}_{\complement U}\mathcal{A}$ of $\mathscr{D}\mathcal{A}$ consisting of complexes whose cohomologies are supported on $\complement U$.

If $W \subseteq X$ is another open affine subset, then the subcategories $\mathscr{D}_{\complement U}\mathcal{A}$ and $\mathscr{D}_{\complement W}\mathcal{A}$ *intersect properly* in the sense of [32, 5.2.3]. Recall that it means that $j_W^* j_{U*} j_U^* \mathcal{F} = 0$ as soon as $j_W^* \mathcal{F} = 0$, what follows, for instance, from [20, Corollaire (1.5.2)] applied to the cartesian diagram of affine morphisms (open embeddings)

$$\begin{array}{ccc} U \cap W & \xrightarrow{j'_W} & U \\ {\scriptstyle j'_U}\big\downarrow & & \big\downarrow{\scriptstyle j_U} \\ W & \xrightarrow[j_W]{} & X \end{array}$$

Therefore, if $X = \bigcup_{i=1}^m U_i$ is an open affine covering of X, then $\{\, \mathscr{D}_{\complement U_i}\mathcal{A} \,\}$ is a cocovering of the triangulated category $\mathscr{D}\mathcal{A}$ as defined in [32, 5.3.3]. If $S \subset \{\, 1, 2, \dots, m \,\}$ does not contain i, $U_S = \bigcup_{j \in S} U_j$, then $\bigcap_{j \in S} \mathscr{D}_{\complement U_j}\mathcal{A} = \mathscr{D}_{\complement U_S}\mathcal{A}$ and the image of $\mathscr{D}_{\complement U_S}\mathcal{A}$ in $\mathscr{D}\mathcal{A}_{U_i}$ coincides with $\mathscr{D}_{U_i \setminus U_S}\mathcal{A}_{U_i}$. There are sections $f_1, f_2, \dots, f_k \in \boldsymbol{A} = \Gamma(U_i, \mathcal{O}_X)$ such that $U_i \setminus U_S = V(f_1, f_2, \dots, f_k)$ as a closed subset of U_i. The following lemma shows that the subcategory $\mathscr{D}_{U_i \setminus U_S}\mathcal{A}_{U_i}$ is compactly generated in $\mathscr{D}\mathcal{A}_{U_i}$.

Lemma 3.15. *Let $\boldsymbol{A}$ be an algebra over a commutative ring $\boldsymbol{O}$ and $\boldsymbol{I} = (f_1, f_2, \dots, f_k)$ be a finitely generated ideal in $\boldsymbol{O}$. Let $K^\bullet(\boldsymbol{I})$ be the corresponding Koszul complex. Denote by $\boldsymbol{A}\text{-}\mathsf{Mod}_{\boldsymbol{I}}$ the full subcategory of $\boldsymbol{A}\text{-}\mathsf{Mod}$ consisting of all modules M such that for every element $a \in M$ there is m such that $\boldsymbol{I}^m a = 0$. Denote by $\mathscr{D}_{\boldsymbol{I}}\boldsymbol{A}$ the full subcategory of $\mathscr{D}\boldsymbol{A}$ consisting of all complexes such that their cohomologies belong to $\boldsymbol{A}\text{-}\mathsf{Mod}_{\boldsymbol{I}}$. Then $\mathscr{D}_{\boldsymbol{I}}\mathcal{A}$ is generated by the complex $K^\bullet_{\boldsymbol{A}}(\boldsymbol{I}) = \boldsymbol{A} \otimes_{\boldsymbol{O}} K^\bullet(\boldsymbol{I})$.*

Proof. Note that $\operatorname{Hom}_{\mathscr{D}\boldsymbol{A}}(K^\bullet_{\boldsymbol{A}}(\boldsymbol{I}), C^\bullet) \simeq \operatorname{Hom}_{\mathscr{D}\boldsymbol{O}}(K^\bullet(\boldsymbol{I}), C^\bullet)$ for every $C^\bullet \in \mathscr{D}\boldsymbol{A}$. If $C^\bullet \in \mathscr{D}_{\boldsymbol{I}}\boldsymbol{A}$ is non-exact, then $\operatorname{Hom}_{\mathscr{D}\boldsymbol{O}}(K^\bullet(\boldsymbol{I}), C^\bullet[n]) \neq 0$ for some n by [32, Proposition 6.6]. It proves the claim. $\square$

Evidently, $K^\bullet_{\boldsymbol{A}}(\boldsymbol{I})$ is compact in $\mathscr{D}\boldsymbol{A}$. So we can now use [32, Theorem 5.15]. It implies that $\mathscr{D}\mathcal{A}$ is compactly generated and a complex $\mathcal{C}^\bullet$ in $\mathscr{D}\mathcal{A}$ is compact if and only if $j_{U_i}^* \mathcal{C}^\bullet$ is compact in $\mathscr{D}\mathcal{A}_{U_i}$ for every $1 \le i \le m$. As U_i is affine, compact complexes in $\mathscr{D}\mathcal{A}_{U_i}$ are just perfect complexes. Therefore, it is true for $\mathscr{D}\mathcal{A}$ too. $\square$

4. Minors

Definition 4.1. Let $(X, \mathcal{B})$ be a non-commutative scheme, $\mathcal{P}$ be a locally projective and locally finitely generated $\mathcal{B}$-module, $\mathcal{A} = (\mathcal{E}nd_{\mathcal{B}}\mathcal{P})^{\mathrm{op}}$. The non-commutative scheme $(X, \mathcal{A})$ is called a *minor* of the non-commutative scheme $(X, \mathcal{B})$.[4]

[4] In the affine case this notion was introduced in [13]. Actually, the main results of this section are just global analogues of those from [13].

In this situation we consider $\mathcal{P}$ as $\mathcal{B}$-$\mathcal{A}$-bimodule (left over $\mathcal{B}$, right over $\mathcal{A}$). Let $\mathcal{P}^\vee = \mathcal{H}om_{\mathcal{B}}(\mathcal{P},\mathcal{B})$. It is an $\mathcal{A}$-$\mathcal{B}$-bimodule, locally projective and locally finitely generated over $\mathcal{B}$. The following statements are evidently local, then they are well-known.

Proposition 4.2. *The natural homomorphism $\mathcal{P} \to \mathcal{H}om_{\mathcal{B}}(\mathcal{P}^\vee,\mathcal{B})$ is an isomorphism. Moreover, $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{B}}\,\mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes_{\mathcal{B}} \mathcal{P}$.*

We consider the following functors:

$$\begin{aligned} \mathsf{F} &= \mathcal{P} \otimes_{\mathcal{A}} \_ : \mathcal{A}\text{-}\mathsf{Mod} \to \mathcal{B}\text{-}\mathsf{Mod}, \\ \mathsf{G} &= \mathcal{H}om_{\mathcal{B}}(\mathcal{P}, \_) : \mathcal{B}\text{-}\mathsf{Mod} \to \mathcal{A}\text{-}\mathsf{Mod}, \\ \mathsf{H} &= \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \_) : \mathcal{A}\text{-}\mathsf{Mod} \to \mathcal{B}\text{-}\mathsf{Mod}. \end{aligned} \tag{4.0.1}$$

Note that G is exact and $\mathsf{G} \simeq \mathcal{P}^\vee \otimes_{\mathcal{B}} \_$, so both (F,G) and (G,H) are adjoint pairs of functors. If the non-commutative scheme $(X,\mathcal{B})$ is noetherian, so is also $(X,\mathcal{A})$ and these functors map coherent sheaves to coherent ones.

Theorem 4.3. (1) *$\mathcal{A}$-$\mathsf{Mod} \simeq \mathcal{B}$-$\mathsf{Mod}/\mathscr{C}$, where $\mathscr{C} = \operatorname{Ker}\mathsf{G} = \mathcal{P}^\perp$ is a bilocalizing subcategory of $\mathcal{B}$-Mod. Thus $\mathcal{A}$-Mod is a bilocalization of $\mathcal{B}$-Mod and G is a bilocalization functor.*

(2) *The natural morphism $\phi : \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}} \to \mathsf{G}\circ\mathsf{F}$ is an isomorphism.*

(2′) *The natural morphism $\phi' : \mathsf{G}\circ\mathsf{H} \to \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}}$ is an isomorphism.*

(3) *The functor F is a full embedding and its essential image is ${}^\perp\mathscr{C}$. So the pair (F,G) induces an equivalence between $\mathcal{A}$-Mod and ${}^\perp\mathscr{C}$.*

(3′) *The functor H is a full embedding and its essential image is $\mathscr{C}^\perp$. So the pair (H,G) induces an equivalence between $\mathcal{A}$-Mod and $\mathscr{C}^\perp$.*

(4) *${}^\perp\mathscr{C}$ coincides with the full subcategory of $\mathcal{B}$-Mod consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an exact sequence $P_1 \to P_0 \to \mathcal{M}_x \to 0$, where P_0, P_1 are multiples of $\mathcal{P}_x$ (i.e. direct sums, maybe infinite, of its copies). We denote this subcategory by $\mathcal{P}$-Mod.*

(4′) *$\mathscr{C}^\perp$ consists with the full subcategory of $\mathcal{B}$-Mod consisting of all modules $\mathcal{M}$ such that there is an exact sequence $0 \to \mathcal{M} \to \mathcal{I}_0 \to \mathcal{I}_1$, where $\mathcal{I}_i \in \mathsf{H}(\mathcal{A}\text{-}\mathsf{Inj})$.*[5] *We denote this subcategory by $\mathcal{P}^{\mathsf{Inj}}$-$\mathsf{Mod}$.*

Proof. Theorem 2.1 and Corollary 2.3 show that it is enough to prove the following statements.

Proposition 4.4. (1) *The natural morphism $\phi : \mathbb{1}_{\mathcal{A}\text{-}\mathsf{Mod}} \to \mathsf{G}\circ\mathsf{F}$ is an isomorphism.*

(2) $\operatorname{Im}\mathsf{F} = \mathcal{P}$-$\mathsf{Mod}$.

(2′) $\operatorname{Im}\mathsf{H} = \mathcal{P}^{\mathsf{Inj}}$-$\mathsf{Mod}$

As the claims (1) and (2) are local, we can suppose that the non-commutative scheme $(X,\mathcal{B})$ is affine, so replace $\mathcal{B}$-Mod by $\boldsymbol{B}$-Mod, where $\boldsymbol{B} = \Gamma(X,\mathcal{B})$. Then $\mathcal{P} = P^\sim$ for some finitely generated projective $\boldsymbol{B}$-module

[5] Note that all $\mathcal{B}$-modules from $\mathsf{H}(\mathcal{A}\text{-}\mathsf{Inj})$ are injective.

and $\mathcal{A} = \boldsymbol{A}^{\sim}$, where $\boldsymbol{A} = (\operatorname{End}_{\boldsymbol{B}} P)^{\mathrm{op}}$. Hence we can also replace $\mathcal{A}$-Mod by $\boldsymbol{A}$-Mod and $\mathcal{P}$-Mod by P-Mod, the full subcategory of $\boldsymbol{B}$-Mod consisting of all modules N such that there is an exact sequence $P_1 \to P_0 \to N \to 0$, where P_i are multiples of P.

Obviously, $\phi(\boldsymbol{A})$ is an isomorphism. Since F and G preserve arbitrary coproducts, $\phi(F)$ is an isomorphism for any free $\boldsymbol{A}$-module F. Let $M \in \mathcal{A}$-Mod. There is an exact sequence $F_1 \to F_0 \to M \to 0$, where F_0, F_1 are free modules, which gives rise to a commutative diagram with exact rows

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, "\phi(F_1)"'] & F_0 \arrow[r] \arrow[d, "\phi(F_0)"'] & M \arrow[r] \arrow[d, "\phi(M)"'] & 0 \\
\mathsf{G}\circ\mathsf{F}(F_1) \arrow[r] & \mathsf{G}\circ\mathsf{F}(F_0) \arrow[r] & \mathsf{G}\circ\mathsf{F}(M) \arrow[r] & 0
\end{tikzcd}$$

As the first two vertical arrows are isomorphisms, so is $\phi(M)$, which proves claim (1). Moreover, we get an exact sequence $\mathsf{F}(F_1) \to \mathsf{F}(F_0) \to \mathsf{F}(M) \to 0$, where $\mathsf{F}(F_i)$ are multiples of $\mathsf{F}(\boldsymbol{A}) = P$. Therefore, $\mathsf{F}(M) \in P$-Mod.

Consider now the natural morphism $\psi : \mathsf{F}\circ\mathsf{G} \to \mathbb{1}_{B\text{-}\mathsf{Mod}}$. This time $\psi(P)$ is an isomorphism. Let now N be a $\boldsymbol{B}$-module such that there is an exact sequence $P_1 \to P_0 \to N \to 0$, where P_i are multiples of P. Then there is a commutative diagram with exact rows

$$\begin{tikzcd}
\mathsf{F}\circ\mathsf{G}(P_1) \arrow[r] \arrow[d, "\psi(P_1)"'] & \mathsf{F}\circ\mathsf{G}(P_0) \arrow[r] \arrow[d, "\psi(P_0)"'] & \mathsf{F}\circ\mathsf{G}(N) \arrow[r] \arrow[d, "\psi(N)"'] & 0 \\
P_1 \arrow[r] & P_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The first two vertical arrows are isomorphisms, so $\psi(N)$ is also an isomorphism. It proves claim (2).

The proof of $(2')$ is quite analogous to that of (2), so we omit it.

Note that the condition $\mathcal{M} \in \mathcal{P}^{\mathsf{Inj}}$-$\mathsf{Mod}$ also turns out to be local, since it means that the natural map $\mathcal{M} \to \mathsf{H}\circ\mathsf{G}(\mathcal{M})$ is an isomorphism. $\square$

Actually, we can describe the kernel of this bilocalization explicitly.

Theorem 4.5. *Let* $\mathcal{I}_{\mathcal{P}} = \operatorname{Im}\{\mu_{\mathcal{P}} : \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}^{\vee} \to \mathcal{B}\}$, *where* $\mu(p \otimes \gamma) = \gamma(p)$. *Then* $\operatorname{Ker}\mathsf{G} = \{\, \mathcal{M} \in \mathcal{B}\text{-}\mathsf{Mod} \mid \mathcal{I}_{\mathcal{P}}\mathcal{M} = 0 \,\} \simeq (\mathcal{B}/\mathcal{I}_{\mathcal{P}})$-$\mathsf{Mod}$.

Proof. Again the statement is local, so we only have to prove it for a ring $\boldsymbol{B}$, a finitely generated projective $\boldsymbol{B}$-module P and the ideal $I_P = \operatorname{Im}\mu_P$. It follows from [9, Proposition VII.3.1] that $I_P P = P$. Therefore, if $f : P \to M$ is non-zero, then $I_P \operatorname{Im} f = \operatorname{Im} f \neq 0$, hence $I_P M \neq 0$. On the contrary, if $I_P M \neq 0$, there is an element $u \in M$, elements $p_i \in P$ and homomorphisms $\gamma_i : P \to B$ such that $\sum_i \gamma_i(p_i)u \neq 0$. Let $\beta : B \to M$ maps 1 to u and $\gamma_i^u = \beta\gamma_i$. Then at least one of the homomorpisms γ_i^u is non-zero. $\square$

The functor G is exact, so it induces a functor $\mathsf{DG} : \mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ mapping a complex $\mathcal{F}^\bullet$ to $\mathsf{G}\mathcal{F}^\bullet$. It is both left and right derived functor of G. We can also consider the left derived functor LF of F and the right derived functor

RH of H, both being functors $\mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$. Obviously, DG maps $\mathscr{D}^\sigma\mathcal{B}$ to $\mathscr{D}^\sigma\mathcal{A}$, where $\sigma \in \{+,-,b\}$, LF maps $\mathscr{D}^-\mathcal{A}$ to $\mathscr{D}^-\mathcal{B}$ and RH maps $\mathscr{D}^+\mathcal{A}$ to $\mathscr{D}^+\mathcal{B}$.

Theorem 4.6. (1) *The functors* $(\mathsf{LF},\mathsf{DG})$ *and* $(\mathsf{DG},\mathsf{RH})$ *form adjoint pairs.*

(2) $\mathscr{D}\mathcal{A} \simeq \mathscr{D}\mathcal{B}/\mathscr{D}_{\mathscr{C}}\mathcal{B}$, *where* $\mathscr{C} = \operatorname{Ker}\mathsf{G} = \mathcal{P}^\perp \simeq (\mathcal{B}/\mathcal{I}_{\mathcal{P}})\text{-}\mathsf{Mod}$. *Moreover,* $\mathscr{D}_{\mathscr{C}}\mathcal{B} = \operatorname{Ker}\mathsf{G}$ *is a bilocalizing subcategory of* $\mathscr{D}\mathcal{B}$, *so* $\mathscr{D}\mathcal{A}$ *is a bilocalization of* $\mathscr{D}\mathcal{B}$ *and* DG *is a bilocalizing functor.*

(3) *The natural map* $\mathbb{1}_{\mathscr{D}\mathcal{A}} \to \mathsf{DG}\circ\mathsf{LF}$ *is an isomorphism.*

(3′) *The natural map* $\mathsf{DG}\circ\mathsf{RH} \to \mathbb{1}_{\mathscr{D}\mathcal{A}}$ *is an isomorphism.*

(4) *The functor* LF *is a full embedding and its essential image is* ${}^\perp(\mathscr{D}_{\mathscr{C}}\mathcal{B})$. *So the pair* $(\mathsf{LF},\mathsf{DG})$ *defines an equivalence* $\mathscr{D}\mathcal{A} \simeq {}^\perp(\mathscr{D}_{\mathscr{C}}\mathcal{B})$.

(4′) *The functor* RH *is a full embedding and its essential image is* $(\mathscr{D}_{\mathscr{C}}\mathcal{B})^\perp$. *So the pair* $(\mathsf{RH},\mathsf{DG})$ *defines an equivalence* $\mathscr{D}\mathcal{A} \simeq (\mathscr{D}_{\mathscr{C}}\mathcal{B})^\perp$.

(5) *The functor* LF *maps* $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ *to* $\mathscr{D}^{\mathrm{c}}\mathcal{B}$.

(6) $(\operatorname{Ker}\mathsf{DG}, \operatorname{Im}\mathsf{LF})$ *as well as* $(\operatorname{Im}\mathsf{RH}, \operatorname{Ker}\mathsf{DG})$ *are semi-orthogonal decompositions of* $\mathscr{D}\mathcal{B}$.
Note that $\operatorname{Im}\mathsf{LF} \simeq \operatorname{Im}\mathsf{RH}$, though these two subcategories usually do not coincide. Both of them are equivalent to $\mathscr{D}\mathcal{A}$.

(7) $\operatorname{Im}\mathsf{LF}$ *coincides with the full subcategory* $\mathscr{D}\underrightarrow{\mathcal{P}}$ *of* $\mathscr{D}\mathcal{B}$ *consisting of complexes quasi-isomorphic to K-flat complexes* $\mathcal{F}^\bullet$ *such that for every* $x \in X$ *and every component* $\mathcal{F}^i$ *the localization* $\mathcal{F}^i_x$ *is a direct limit of modules from* $\operatorname{add}\mathcal{P}_x$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^-$.

(7p) *If* $\mathcal{A}$ *and* $\mathcal{B}$ *have enough locally projective modules (for instance, if* X *is quasi-projective),* $\operatorname{Im}\mathsf{LF}$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}$ *of* $\mathscr{D}\mathcal{B}$ *consisting of complexes quasi-isomorphic to K-flat complexes* $\mathcal{F}^\bullet$ *such that* $\mathcal{F}^i_x \in \operatorname{Add}\mathcal{P}_x$ *for every* $i \in \mathbb{Z}$ *and every point* $x \in X$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^-$.

(7′) $\operatorname{Im}\mathsf{RH}$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}^{\mathsf{Inj}}$ *of* $\mathscr{D}\mathcal{B}$ *consisting of complexes quasi-isomorphic to K-injective complexes consisting of modules from* $\mathsf{H}(\mathcal{A}\text{-}\mathsf{Inj})$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^+$.

Note that the condition in (7′) can also be verified locally at every point $x \in X$.

Proof. (1) Let $\mathcal{F}^\bullet$ be a K-flat replica of $\mathcal{M}^\bullet \in \mathscr{D}\mathcal{A}$ and $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{N}^\bullet \in \mathscr{D}\mathcal{B}$. Then $\mathsf{LF}\mathcal{M}^\bullet = \mathsf{F}\mathcal{F}^\bullet$ and $\mathsf{DG}\mathcal{N}^\bullet = \mathsf{G}\mathcal{I}^\bullet$. As $\mathcal{P} \in \mathsf{lp}\,\mathcal{B}$, the complex $\mathsf{F}\mathcal{F}^\bullet$ is K-flat and the complex $\mathsf{G}\mathcal{I}^\bullet$ is K-injective. By Proposition 3.12 (2),

$$\operatorname{RHom}_{\mathcal{B}}(\mathsf{F}\mathcal{F}^\bullet,\mathcal{I}^\bullet) = \operatorname{Hom}^\bullet_{\mathcal{B}}(\mathsf{F}\mathcal{F}^\bullet,\mathcal{I}^\bullet) \simeq \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathcal{F}^\bullet,\mathsf{G}\mathcal{I}^\bullet) = \operatorname{RHom}_{\mathcal{A}}(\mathcal{F}^\bullet,\mathsf{G}\mathcal{I}^\bullet).$$

Taking zero cohomologies, we obtain that

$$\operatorname{Hom}_{\mathcal{B}}(\mathsf{F}\mathcal{F}^\bullet,\mathcal{I}^\bullet) \simeq \operatorname{Hom}_{\mathcal{A}}(\mathcal{F}^\bullet,\mathsf{G}\mathcal{I}^\bullet).$$

Choose now a K-flat replica $\mathcal{G}^\bullet$ of $\mathcal{N}^\bullet$ and a K-injective resolution $\mathcal{J}^\bullet$ of $\mathcal{M}^\bullet$. Then $\mathsf{DG}\mathcal{N}^\bullet = \mathsf{G}\mathcal{G}^\bullet$ and $\mathsf{RH}\mathcal{M}^\bullet = \mathsf{H}\mathcal{J}^\bullet$. By [33, Proposition 5.14], $\mathsf{H}\mathcal{J}^\bullet$ is weakly K-injective. By Proposition 3.12 (2) and [33, Proposition 6.1],

$$\mathrm{RHom}_{\mathcal{A}}(\mathsf{G}\mathcal{G}^\bullet, \mathcal{J}^\bullet) = \mathrm{Hom}^\bullet_{\mathcal{A}}(\mathsf{G}\mathcal{G}^\bullet, \mathcal{J}^\bullet) \simeq \\ \mathrm{Hom}^\bullet_{\mathcal{B}}(\mathcal{G}^\bullet, \mathsf{H}\mathcal{J}^\bullet) = \mathrm{RHom}_{\mathcal{B}}(\mathcal{G}^\bullet, \mathsf{H}\mathcal{J}^\bullet).$$

Taking zero cohomologies, we obtain that

$$\mathrm{Hom}_{\mathcal{A}}(\mathsf{G}\mathcal{G}^\bullet, \mathcal{J}^\bullet) \simeq \mathrm{Hom}_{\mathcal{B}}(\mathcal{G}^\bullet, \mathsf{H}\mathcal{J}^\bullet)$$

The statements (3) and (3′) follow from the statements (2) and (2′) of Theorem 4.3. Then the statements (2),(4) and (4′) follow from Theorem 2.1 and Corollary 2.3.

(5) As the right adjoint DG of LF preserves arbitrary coproducts, LF maps compact objects to compact ones.

(6) follows from Theorem 2.5.

(7) The construction of [1, Proposition 1.1] gives for any complex $\mathcal{M}^\bullet \in \mathscr{D}\mathscr{A}$ a quasi-isomorphic K-flat complex $\mathcal{F}^\bullet$ such that all its components $\mathcal{F}^i$ are flat. Moreover, $\mathcal{F}^\bullet$ is left bounded if so is $\mathcal{M}^\bullet$. By [6, Ch. X, § 1, Théorème 1], $\mathcal{F}^i_x \simeq \varinjlim \mathcal{L}^i_n$, where $\mathcal{L}^i_n$ are projective finitely generated $\mathcal{A}_x$-modules, hence belong to $\mathrm{add}\,\mathcal{A}_x$. Then $\mathsf{LF}\mathcal{M}^\bullet \simeq \mathsf{F}\mathcal{F}^\bullet$. As F preserves direct limits and $\mathsf{F}\mathcal{A} \simeq \mathcal{P}$, $\mathsf{F}\mathcal{F}^i_x \simeq \varinjlim \mathsf{F}\mathcal{L}^i_n$ and $\mathsf{F}\mathcal{L}^i_n \in \mathrm{add}\,\mathcal{P}_x$. Hence $\mathcal{M}^\bullet \in \mathscr{D}\underrightarrow{\mathcal{P}}$.

On the contrary, let $\mathcal{N}^\bullet \in \mathscr{D}\underrightarrow{\mathcal{P}}$. We can suppose that it is K-flat and for every $i \in \mathbb{Z}$ and every $x \in X$ we can present $\mathcal{N}^i_x$ as $\varinjlim \mathcal{N}^i_n$, where $\mathcal{N}^i_n \in \mathrm{add}\,\mathcal{P}_x$. Then the complex $\mathsf{G}\mathcal{N}^\bullet$ is also K-flat [33, Proposition 5.4], so $\mathsf{LF} \circ \mathsf{DG}(\mathcal{N}^\bullet) \simeq \mathsf{FG}(\mathcal{N}^\bullet)$. As the natural map $\mathsf{FG}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism, the same is true for all modules $\mathcal{N}^i_n$, hence also for $\mathcal{N}^i_x$. Therefore, the natural map $\mathsf{LF} \circ \mathsf{DG}(\mathcal{N}) \to \mathcal{N}$ is an isomorphism.

The proof of (7p) is quite analogous to the proof of (7), taking into account that in this situation every complex is quasi-isomorphic to a K-flat complex of locally projective modules. The proof of (7′) is also analogous to that of (7). □

There is one special case when the category Ker DG can be described more precisely.

Theorem 4.7. *Suppose that the ideal $\mathcal{I}_{\mathcal{P}}$ is flat as a right $\mathcal{B}$-module. Then* $\mathrm{Ker}\,\mathsf{DG} \simeq \mathscr{D}(\mathcal{B}/\mathcal{I}_{\mathcal{P}})$.

Proof. Let $\mathcal{I} = \mathcal{I}_{\mathcal{P}}$, $\mathcal{Q} = \mathcal{B}/\mathcal{I}$. One easily sees that $\mathcal{I}^2 = \mathcal{I}$. We identify $\mathscr{D}\mathcal{Q}$ with the full triangulated subcategory of $\mathscr{D}\mathcal{B}$, obviously contained in $\mathrm{Ker}\,\mathsf{DG}$. Let $\mathcal{F}^\bullet \in \mathrm{Ker}\,\mathsf{DG}$, i.e. its cohomologies are indeed $\mathcal{Q}$-modules. We can suppose that $\mathcal{F}^\bullet$ is K-flat. Tensoring it with the exact sequence $0 \to \mathcal{I} \to \mathcal{B} \to \mathcal{Q} \to 0$, we obtain an exact sequence of complexes $0 \to \mathcal{I} \otimes_{\mathcal{B}} \mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{F}^\bullet \to 0$. Since $\mathcal{I}$ is flat, $H^\bullet(\mathcal{I} \otimes_{\mathcal{B}} \mathcal{F}) \simeq \mathcal{I} \otimes_{\mathcal{B}} H^\bullet(\mathcal{F}^\bullet)$. Note that $\mathcal{I} \otimes_{\mathcal{B}} \mathcal{Q} \simeq \mathcal{I}/\mathcal{I}^2 = 0$, whence $\mathcal{I} \otimes_{\mathcal{B}} \mathcal{M} = 0$ for any $\mathcal{Q}$-module.

Therefore, $H^\bullet(\mathcal{I} \otimes_{\mathcal{B}} \mathcal{F}^\bullet) = 0$, hence $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{Q} \otimes_{\mathcal{B}} \mathcal{F}^\bullet$, which is in $\mathscr{D}\mathcal{Q}$. □

Example 4.8. An important special case of minors appears as the *endomorphism construction.* Let $\mathcal{A}$ be a non-commutative scheme, $\mathcal{F}$ be a coherent $\mathcal{A}$-module and $\mathcal{A}_{\mathcal{F}} = \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \oplus \mathcal{F})^{\mathrm{op}}$. Then $\mathcal{A}_{\mathcal{F}}$ is identified with the algebra of matrices

$$\mathcal{A}_{\mathcal{F}} = \begin{pmatrix} \mathcal{A} & \mathcal{F} \\ \mathcal{F}' & \mathcal{E} \end{pmatrix}$$

where $\mathcal{F}' = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$ and $\mathcal{E} = (\mathcal{E}nd_{\mathcal{A}} \mathcal{F})^{\mathrm{op}}$. If $\mathcal{P}_{\mathcal{F}} = \left(\begin{smallmatrix} \mathcal{A} \\ \mathcal{F}' \end{smallmatrix}\right)$ considered as $\mathcal{A}_{\mathcal{F}}$-module, then $\mathcal{A} \simeq (\mathcal{E}nd_{\mathcal{A}_{\mathcal{F}}} \mathcal{P}_{\mathcal{F}})^{\mathrm{op}}$, so $\mathcal{A}$ is a minor of $\mathcal{A}_{\mathcal{F}}$ and the categories $\mathcal{A}\text{-}\mathsf{Mod}$ and $\mathscr{D}\mathcal{A}$ are bilocalizations, respectively, of $\mathcal{A}_{\mathcal{F}}\text{-}\mathsf{Mod}$ and $\mathscr{D}\mathcal{A}_{\mathcal{F}}$. The corresponding functors are

$$\begin{aligned} \mathsf{F}_{\mathcal{F}} &= \mathcal{P}_{\mathcal{F}} \otimes_{\mathcal{A}} \_\,, \\ \mathsf{G}_{\mathcal{F}} &= \mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{P}_{\mathcal{F}}, \_), \\ \mathsf{H}_{\mathcal{F}} &= \mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{P}_{\mathcal{F}}, \_). \end{aligned}$$

Note that $\mathcal{P}_{\mathcal{F}}^\vee \simeq \begin{pmatrix} \mathcal{A} & \mathcal{F} \end{pmatrix}$ as right $\mathcal{A}_{\mathcal{F}}$-module and, by the construction, $\mathcal{P}_{\mathcal{F}} \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{P}_{\mathcal{F}}^\vee, \mathcal{A})$. Theorem 4.5 then implies that the kernel $\mathscr{C}$ of the functor $\mathsf{G}_{\mathcal{F}} : \mathcal{A}_{\mathcal{F}}\text{-}\mathsf{Mod} \to \mathcal{A}\text{-}\mathsf{Mod}$ is equivalent to $\mathcal{E}/\mathcal{I}_{\mathcal{F}}\text{-}\mathsf{Mod}$, where $\mathcal{I}_{\mathcal{F}}$ is the image of the natural map $\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{E}$.

We consider an application of minors to global dimensions and semi-orthogonal decompositions. Let $(X, \mathcal{B})$ be a non-commutative scheme, $\mathcal{M}$ be a $\mathcal{B}$-module. We call $\sup \left\{ i \mid \mathcal{E}xt^i_{\mathcal{B}}(\mathcal{M}, \_) \neq 0 \right\}$ the *local projective dimension* of the $\mathcal{B}$-module $\mathcal{M}$ and denote it by $\mathrm{lp.dim}_{\mathcal{B}} \mathcal{M}$. If $(X, \mathcal{B})$ is noetherian and $\mathcal{M}$ is coherent, then $\mathrm{lp.dim}_{\mathcal{B}} \mathcal{M} = \sup \left\{ \mathrm{pr.dim}_{\mathcal{B}_x} \mathcal{M}_x \mid x \in X \right\}$.

Lemma 4.9. *Let $(X, \mathcal{B})$ be a non-commutative scheme, $\mathcal{P}$ be a locally projective and locally finitely generated $\mathcal{B}$-module, $\mathcal{A} = (\mathcal{E}nd_{\mathcal{B}} \mathcal{P})^{\mathrm{op}}$ and $\bar{\mathcal{B}} = \mathcal{B}/\mathcal{I}_{\mathcal{P}}$. Suppose that $\mathcal{P}$ is flat as right $\mathcal{A}$-module,*

$$\begin{aligned} \mathrm{lp.dim}_{\mathcal{B}} \mathcal{I}_{\mathcal{P}} &= d, \\ \mathrm{gl.dim}\, \mathcal{A} &= n, \\ \mathrm{gl.dim}\, \bar{\mathcal{B}} &= m. \end{aligned}$$

Then $\mathrm{gl.dim}\, \mathcal{B} \leq \max \{ m + d + 2,\, n \}$.

Proof. Let $\bar{\mathcal{B}} = \mathcal{B}/\mathcal{I}_{\mathcal{P}}$. Then $\mathrm{lp.dim}_{\mathcal{B}} \bar{\mathcal{B}} = d + 1$. From the spectral sequence $\mathrm{Ext}^p_{\bar{\mathcal{B}}}(\mathcal{M}, \mathcal{E}xt^q_{\mathcal{B}}(\bar{\mathcal{B}}, \_)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{B}}(\mathcal{M}, \_)$ it follows that $\mathrm{pr.dim}_{\mathcal{B}} \mathcal{M} \leq m + d + 1$ for every $\bar{\mathcal{B}}$-module $\mathcal{M}$. Consider the functors $\mathsf{G} = \mathrm{Hom}_{\mathcal{B}}(\mathcal{P}, \_)$ and $\mathsf{F} = \mathcal{P} \otimes_{\mathcal{A}} \_$. Since the morphism $\mathsf{GFG} \to \mathsf{G}$, arising from the adjunction, is an isomorphism, the kernel and the cokernel of the natural map $\alpha : \mathsf{F}\mathsf{G}\mathcal{M} \to \mathcal{M}$ are annihilated by G, so are actually $\bar{\mathcal{B}}$-modules. It implies that $\mathrm{Ext}^i_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Ext}^i_{\mathcal{B}}(\mathsf{FG}\mathcal{M}, \mathcal{N})$ if $i > m + d + 2$, so $\mathrm{pr.dim}_{\mathcal{B}} \mathcal{M} \leq \max \{ m + d + 2,\, \mathrm{pr.dim}_{\mathcal{B}} \mathsf{FG}\mathcal{M} \}$. As both functors F and G are exact, $\mathrm{Ext}^i_{\mathcal{B}}(\mathsf{F}\_, \_) \simeq \mathrm{Ext}^i_{\mathcal{A}}(\_, \mathsf{G}\_)$, so $\mathrm{pr.dim}_{\mathcal{B}} \mathsf{FG}\mathcal{M} \leq n$. □

Definition 4.10. (1) Let $(X,\mathcal{B})$ and $(X,\mathcal{A})$ be two non-commutative schemes. A *relating chain* between $\mathcal{B}$ and $\mathcal{A}$ is a sequence $(\mathcal{B}_1,\mathcal{P}_1,\mathcal{B}_2,\mathcal{P}_2,\dots,\mathcal{P}_r,\mathcal{B}_{r+1})$, where $\mathcal{B}_1=\mathcal{B}$, $\mathcal{B}_{r+1}=\mathcal{A}$, every $\mathcal{P}_i$ $(1\le i\le r)$ is a locally projective and locally finitely generated $\mathcal{B}_i$-module which is also flat as right $\mathcal{A}_i$-module, where $\mathcal{A}_i=(\mathcal{E}nd_{\mathcal{B}_i}\mathcal{P}_i)^{\mathrm{op}}$, and $\mathcal{B}_{i+1}=\mathcal{B}_i/\mathcal{I}_{\mathcal{P}_i}$ for $1\le i\le r$.

(2) The relating chain is said to be *flat* if, for every $1\le i\le r$, $\mathcal{I}_{\mathcal{P}_i}$ is flat as right $\mathcal{B}_i$-module. Note that it is the case if the natural map $\mathcal{P}_i\otimes_{\mathcal{A}_i}\mathcal{P}_i^\vee\to\mathcal{B}_i$ is a monomorphism.

(3) The relating chain is said to be *pre-heredity* if, for every $1\le i\le r$, $\mathcal{I}_{\mathcal{P}_i}$ is locally projective as left $\mathcal{B}$-module. If it is both pre-heredity and flat, it is said to be *heredity*.

(4) If the relating chain is heredity and all non-commutative schemes $\mathcal{A}_i$ are hereditary, i.e. $\mathrm{gl.dim}\,\mathcal{A}_i\le 1$, we say that the non-commutative scheme $\mathcal{B}$ is *quasi-hereditary* of level r. (Thus quasi-hereditary of level 0 means hereditary).

We fix a relating chain $(\mathcal{B}_1,\mathcal{P}_1,\mathcal{B}_2,\mathcal{P}_2,\dots,\mathcal{P}_r,\mathcal{B}_{r+1})$ between $\mathcal{B}$ and $\mathcal{A}$ and keep the notations of Definition 4.10 (1).

Corollary 4.11. *Let* $\mathrm{gl.dim}\,\mathcal{A}_i\le n$ *and* $\mathrm{lp.dim}_{\mathcal{B}_i}\mathcal{I}_{\mathcal{P}_i}\le d$ *for all* $1\le i\le r$. *Then* $\mathrm{gl.dim}\,\mathcal{B}\le r(d+2)+\max\{\,\mathrm{gl.dim}\,\mathcal{A},\ n-d-2\,\}$. *If this relating chain is pre-heredity, then* $\mathrm{gl.dim}\,\mathcal{B}\le\mathrm{gl.dim}\,\mathcal{A}+2r$.

Using Theorem 4.6 (6), Theorem 4.7 and induction, we also get the following result.

Corollary 4.12. *If this relating chain is flat, there are semi-orthogonal decompositions* $(\mathscr{T},\mathscr{T}_r,\dots,\mathscr{T}_1)$ *and* $(\mathscr{T}_1',\mathscr{T}_2',\dots,\mathscr{T}_r',\mathscr{T})$ *of* $\mathscr{D}\mathcal{B}$ *such that* $\mathscr{T}_i\simeq\mathscr{T}_i'\simeq\mathscr{D}\mathcal{A}_i$ $(1\le i\le r)$ *and* $\mathscr{T}\simeq\mathscr{D}\mathcal{A}$.

Note that, as a rule, $\mathcal{T}_i\ne\mathcal{T}_i'$.

Corollary 4.13. *If a non-commutative scheme* $\mathcal{B}$ *is quasi-hereditary of level* r, *then* $\mathrm{gl.dim}\,\mathcal{B}\le 2r+1$ *and there are semi-orthogonal decompositions* $(\mathscr{T},\mathscr{T}_r,\dots,\mathscr{T}_1)$ *and* $(\mathscr{T}_1',\mathscr{T}_2',\dots,\mathscr{T}_r',\mathscr{T})$ *of* $\mathscr{D}\mathcal{B}$ *such that* $\mathscr{T}_i\simeq\mathscr{T}_i'$ $(1\le i\le r)$, *as well as* $\mathscr{T}$, *is equivalent to the derived category of a hereditary non-commutative scheme.*

Remark 4.14. Suppose that $(X,\mathcal{B})$ is affine: $X=\mathrm{Spec}\,\boldsymbol{R}$ and $\mathcal{B}=\boldsymbol{B}^\sim$.

(1) If $\boldsymbol{B}$ is semiprimary, then $\mathcal{B}$ is quasi-hereditary with respect to our definition if and only if $\boldsymbol{B}$ is quasi-hereditary in the classical sense of [10, 12].

(2) If $\boldsymbol{R}$ is a discrete valuation ring and $\boldsymbol{B}$ is an $\boldsymbol{R}$-order in a separable algebra, then $\mathcal{B}$ is quasi-hereditary with respect to our definition if and only if $\boldsymbol{B}$ is quasi-hereditary in the sense of [23].

Example 4.15. Consider the endomorphism construction of Example 4.8. Suppose that $\mathcal{F}$ is flat as right $\mathcal{E}$-module, $\mathcal{F}'$ is locally projective as left

$\mathcal{E}$-module and the natural map $\mu_{\mathcal{F}} : \mathcal{F} \otimes_{\mathcal{E}} \mathcal{F}' \to \mathcal{A}$ is a monomorphism. Let $\tilde{\mathcal{P}} = \binom{\mathcal{F}}{\mathcal{E}}$ and $\bar{\mathcal{A}} = \mathcal{A}/\operatorname{Im}\mu_{\mathcal{F}}$. Then one can easily verify that $(\mathcal{A}_{\mathcal{F}}, \tilde{\mathcal{P}}, \bar{\mathcal{A}})$ is a heredity relating chain. Therefore, if both $\mathcal{E}$ and $\bar{\mathcal{A}}$ are quasi-hereditary, so is $\mathcal{A}_{\mathcal{F}}$.

5. Strongly Gorenstein schemes

In this section we only consider noetherian non-commutative schemes.

Definition 5.1. Let $(X, \mathcal{A})$ be a noetherian non-commutative scheme. We call it *strongly Gorenstein* if X is equidimensional, $\mathcal{A}$ is Cohen-Macaulay as $\mathcal{O}_X$-module and $\operatorname{inj.dim}_{\mathcal{A}} \mathcal{A} = \dim X$.[6]

Recall that an $\mathcal{A}$-module $\mathcal{M}$ is injective if and only if $\mathcal{A}_x$-modules $\mathcal{M}_x$ are injective for all $x \in X_{\text{cl}}$ (the proof from [21, Proposition 7.17] remains valid in non-commutative situation too). We need some basic facts about injective dimension for non-commutative rings. Now $\boldsymbol{R}$ denotes a noetherian commutative local ring with the maximal ideal $\mathfrak{m}$ and the residue field $\Bbbk = \boldsymbol{R}/\mathfrak{m}$, $\boldsymbol{A}$ denotes an $\boldsymbol{R}$-algebra finitely generated as $\boldsymbol{R}$-module. Let also $\mathfrak{r} = \operatorname{rad}\boldsymbol{A}$ and $\bar{\boldsymbol{A}} = \boldsymbol{A}/\mathfrak{r}$. As usually, for every ideal $I \subseteq \boldsymbol{R}$ we denote by $V(I)$ the set of prime ideals containing I.

Theorem 5.2. $\operatorname{inj.dim} M = \sup \left\{ i \mid \operatorname{Ext}^i_{\boldsymbol{A}}(\bar{\boldsymbol{A}}, M) \neq 0 \right\}$.

Just as in [7, Proposition 3.1.14], this theorem is an immediate consequence of the following lemma.

Lemma 5.3. *Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of $\boldsymbol{R}$, M be a noetherian $\boldsymbol{R}$-module. Suppose that $\operatorname{Ext}^i_{\boldsymbol{A}}(N, M) = 0$ for any noetherian $\boldsymbol{A}$-module N such that $V(\operatorname{ann}_{\boldsymbol{R}} N) \subset V(\mathfrak{p})$ and $i > m$. Then also $\operatorname{Ext}^i_{\boldsymbol{A}}(N, M) = 0$ for any noetherian $\boldsymbol{A}$-module N such that $V(\operatorname{ann}_{\boldsymbol{R}} N) \subseteq V(\mathfrak{p})$ and $i > m$.*

Proof. Suppose that the condition is satisfied and let $V(\operatorname{ann}_{\boldsymbol{R}} N) \subseteq V(\mathfrak{p})$. If $\mathfrak{q} \in \operatorname{Ass} N$ and $\mathfrak{q} \neq \mathfrak{p}$, there is a submodule $N' \subseteq N$ such that $\mathfrak{q}N' = 0$. Therefore, $\operatorname{Ext}^i_{\boldsymbol{A}}(N', M) = 0$ for $i > m$ and we only have to prove that $\operatorname{Ext}^i_{\boldsymbol{A}}(N/N', M) = 0$ for $i > m$. Thus we can suppose that $\operatorname{Ass} N = \{\mathfrak{p}\}$. Let $a \in \mathfrak{m} \setminus \mathfrak{p}$. Then a is non-zero-divisor on N, i.e. we have the exact sequence $0 \to N \xrightarrow{a} N \to N/aN \to 0$. It gives and exact sequence

$$\operatorname{Ext}^i_{\boldsymbol{A}}(N, M) \xrightarrow{a} \operatorname{Ext}^i_{\boldsymbol{A}}(N, M) \to \operatorname{Ext}^{i+1}_{\boldsymbol{A}}(N/aN, M).$$

Obviously, $\operatorname{ann}_{\boldsymbol{R}} N/aN \supset \mathfrak{p}$, so the last term is 0 if $i > m$. Therefore, $a\operatorname{Ext}^i_{\boldsymbol{A}}(N, M) = \operatorname{Ext}^i_{\boldsymbol{A}}(N, M)$ and $\operatorname{Ext}^i_{\boldsymbol{A}}(N, M) = 0$ by Nakayama's Lemma. $\square$

[6] We do not know whether the last condition implies the Cohen-Macaulay property, as it is in the commutative case.

Corollary 5.4. *Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module. Then*

$$\begin{aligned}\operatorname{inj.dim}_{\mathcal{A}} \mathcal{M} &= \sup \left\{ i \mid \operatorname{Ext}^i_{\mathcal{A}}(\mathcal{A}(x), \mathcal{M}) \neq 0 \text{ for some } x \in X_{\mathrm{cl}} \right\} = \\ &= \sup \left\{ \operatorname{inj.dim}_{\mathcal{A}_x} \mathcal{M}_x \mid x \in X_{\mathrm{cl}} \right\}.\end{aligned}$$

Here $\mathcal{A}(x)$ denotes $\mathcal{A} \otimes_{\mathcal{O}_X} \Bbbk(x)$.

Corollary 5.5.

$$\begin{aligned}\operatorname{gl.dim} \mathcal{A} &= \sup \{ \operatorname{pr.dim}_{\mathcal{A}} \mathcal{A}(x) \mid x \in X_{\mathrm{cl}} \} = \\ &= \sup \left\{ i \mid \operatorname{Ext}^i_{\mathcal{A}}(\mathcal{A}(x), \mathcal{A}(x)) \neq 0 \text{ for some } x \in X_{\mathrm{cl}} \right\} = \\ &= \sup \{ \operatorname{gl.dim} \mathcal{A}_x \mid x \in X_{\mathrm{cl}} \}.\end{aligned}$$

Lemma 5.6. *Let M be a noetherian $\boldsymbol{A}$-module. If an element $a \in \boldsymbol{R}$ is non-zero-divisor both on $\boldsymbol{A}$ and on M, then* $\operatorname{inj.dim}_{\boldsymbol{A}} M = \operatorname{inj.dim}_{\boldsymbol{A}/a\boldsymbol{A}} M/aM$.

Proof. It just repeats that of [7, Corollary 3.1.15]. □

Corollary 5.7. *Let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be an $\boldsymbol{A}$-sequence in $\mathfrak{m}$. Then $\boldsymbol{A}$ is strongly Gorenstein if and only if so is $\boldsymbol{A}/\mathbf{a}\boldsymbol{A}$.*

Corollary 5.8. *$\mathcal{A}$ is strongly Gorenstein if and only if so is $\mathcal{A}^{\mathrm{op}}$.*

Proof. The claim is local, so we can replace $\mathcal{A}$ by $\boldsymbol{A}$. Corollary 5.7 reduces the proof to the case when $\operatorname{Kr.dim} \boldsymbol{R} = 0$, i.e. $\boldsymbol{A}$ is just an artinian algebra. Then it is well-known [4, Proposition IV.3.1]. □

For a noetherian non-commutative scheme $(X, \mathcal{A})$ we denote by $\mathrm{CM}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of such modules $\mathcal{M}$ that $\mathcal{M}_x$ is a maximal Cohen-Macaulay module over $\mathcal{O}_{X,x}$ for every point $x \in X$. The following results can be proved just as in the commutative case (see [7, Section 3.3]).

Theorem 5.9. *Let $(X, \mathcal{A})$ be a strongly Gorenstein non-commutative scheme, $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$.*

(1) *$\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) = 0$ for $i \neq 0$.*
(2) *The natural map $\mathcal{M} \to \mathcal{H}om_{\mathcal{A}}(\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{A}), \mathcal{A})$ is an isomorphism.*

Thus the functor $^* : \mathcal{M} \mapsto \mathcal{M}^* = \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ gives an exact duality between the categories $\mathrm{CM}\,\mathcal{A}$ and $\mathrm{CM}\,\mathcal{A}^{\mathrm{op}}$.

Let now $(X, \mathcal{A})$ be a strongly Gorenstein non-commutative scheme, $\mathcal{F} \in \mathrm{CM}\,\mathcal{A}$. Consider the endomorphism construction described in Example 4.8. Theorem 5.9 implies that the natural map $\phi(\mathcal{M}) : \mathsf{F}_{\mathcal{F}}\mathcal{M} \to \mathsf{H}_{\mathcal{F}}\mathcal{M}$ is an isomorphism for $\mathcal{M} = \mathcal{A}$, hence an isomorphism for any $\mathcal{M} \in \mathsf{lp}\,\mathcal{A}$.

Theorem 5.10. *Let $(X, \mathcal{A})$ be strongly Gorenstein and contain enough locally projective modules, $\mathcal{F} \in \mathrm{CM}\,\mathcal{A}$. Then the restrictions of the functors $\mathsf{LF}_{\mathcal{F}}$ and $\mathsf{RH}_{\mathcal{F}}$ onto the subcategory $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ are isomorphic. Thus the restriction of $\mathsf{LF}_{\mathcal{F}}$ onto $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ is both left and right adjoint to the bilocalization functor $\mathsf{DG}_{\mathcal{F}}$.*

Proof. As $\mathcal{A}$ has enough locally projective modules, any complex from $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ is quasi-isomorphic to a finite complex $\mathcal{C}^\bullet$ such that all $\mathcal{C}^i$ are from $\mathsf{lp}\,\mathcal{A}$. Then $\mathsf{LF}_{\mathcal{F}}\mathcal{C}^\bullet = \mathsf{F}_{\mathcal{F}}\mathcal{C}^\bullet$. On the other hand, by Theorem 5.9, $\mathsf{R}^k\mathsf{H}_{\mathcal{F}}\mathcal{C}^i = \mathcal{E}xt^k_{\mathcal{A}}(\mathcal{P}_{\mathcal{F}},\mathcal{C}^i) = 0$ for $k \neq 0$. Therefore, $\mathsf{RH}_{\mathcal{F}}\mathcal{C}^\bullet = \mathsf{H}_{\mathcal{F}}\mathcal{C}^\bullet \simeq \mathsf{F}_{\mathcal{F}}\mathcal{C}^\bullet$. □

6. Non-commutative curves

6.1. Generalities.

Definition 6.1. A *non-commutative curve* is a reduced non-commutative scheme $(X,\mathcal{A})$ such that X is an excellent curve (equidimensional reduced noetherian scheme of dimension 1) and $\mathcal{A}$ is coherent and torsion free as $\mathcal{O}_X$-module.

As X is excellent, then $\hat{\mathcal{A}}_x$, the $\mathfrak{m}_x$-adic completion of $\mathcal{A}_x$, is also reduced (has no nilpotent ideals). Therefore, for the local study of non-commutative curves we can use the usual results from the books [11, 30]. We denote by $\mathcal{K} = \mathcal{K}(X)$ the sheaf of full rings of fractions of $\mathcal{O}_X$ and write $\mathcal{K}\mathcal{M}$ instead of $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$ for any $\mathcal{O}_X$-module $\mathcal{M}$. In particular, $\mathcal{K}\mathcal{A}$ is a $\mathcal{K}$-algebra. The sheaves $\mathcal{K}\mathcal{M}$ are locally constant; the stalks of $\mathcal{K}$ and $\mathcal{K}\mathcal{A}$ are semi-simple rings. The *torsion part* $\operatorname{tors}\mathcal{M}$ of $\mathcal{M}$ is defined as the kernel of the natural map $\mathcal{M} \to \mathcal{K}\mathcal{M}$. We say that a coherent $\mathcal{A}$-module $\mathcal{M}$ is *torsion free* if $\operatorname{tors}\mathcal{M} = 0$, and we say that $\mathcal{M}$ is *torsion* if $\mathcal{K}\mathcal{M} = 0$. Note that $\operatorname{tors}\mathcal{M}$ is torsion and $\mathcal{M}/\operatorname{tors}\mathcal{M}$ is torsion free. We denote by $\mathsf{tors}\,\mathcal{A}$ and $\mathsf{tf}\,\mathcal{A}$ respectively the full subcategories of $\mathcal{A}$-mod consisting of torsion and of torsion free modules. We always consider a torsion free module $\mathcal{M}$ as a submodule of $\mathcal{K}\mathcal{M}$. In particular, we identify $\mathcal{M}_x$ with its natural image in $\mathcal{K}\mathcal{M}_x$. Note that for every submodule $\mathcal{N} \subseteq \mathcal{K}\mathcal{M}$ there is a natural embedding $\mathcal{K}\mathcal{N} \to \mathcal{K}\mathcal{M}$ and we identify $\mathcal{K}\mathcal{N}$ with the image of this embedding. A non-commutative curve $(X,\mathcal{A}')$ is said to be an *over-ring* of a non-commutative curve $(X,\mathcal{A})$ if $\mathcal{A} \subseteq \mathcal{A}' \subset \mathcal{K}\mathcal{A}$. Then $\mathcal{A}'$ is naturally considered as a coherent $\mathcal{A}$-module. The non-commutative curve $(X,\mathcal{A})$ is said to be *normal* if it has no proper over-rings. Since X is excellent and $\mathcal{A}$ is reduced, the set $\{\, x \in X \mid \mathcal{A}_x \text{ is not normal} \,\}$ is finite. Then it follows from [14] that the set of over-rings of $\mathcal{A}$ satisfies the maximality condition: there are no infinite strictly ascending chains of over-rings of $\mathcal{A}$.

Coherent torsion free $\mathcal{A}$-modules, in particular, over-rings of $\mathcal{A}$ can be constructed locally.

Lemma 6.2. *Let $\mathcal{M}$ be a torsion free coherent $\mathcal{A}$-module.*

(1) *If $\mathcal{N}$ is a coherent $\mathcal{A}$-submodule of $\mathcal{K}\mathcal{M}$ such that $\mathcal{K}\mathcal{N} = \mathcal{K}\mathcal{M}$, then $\mathcal{N}_x = \mathcal{M}_x$ for almost all $x \in X$.*

(2) *Let $S \subset X_{\mathrm{cl}}$ be a finite set and for every $x \in S$ a finitely generated $\mathcal{A}_x$-submodule $N_x \subset \mathcal{K}\mathcal{M}_x$ is given such that $\mathcal{K}N_x = \mathcal{K}\mathcal{M}_x$. Then there is a unique $\mathcal{A}$-submodule $\mathcal{N} \subset \mathcal{K}\mathcal{M}$ such that $\mathcal{N}_x = N_x$ for every $x \in S$ and $\mathcal{N}_x = \mathcal{M}_x$ for all $x \notin S$.*

(3) *If $\mathcal{M} = \mathcal{A}$ and all N_x in the preceding item are rings, then $\mathcal{N}$ is a subalgebra of $\mathcal{KA}$, so $(X, \mathcal{N})$ is also a non-commutative curve and if $N_x \supseteq \mathcal{A}_x$ for all $x \in S$, $(X, \mathcal{N})$ is an over-ring of $(X, \mathcal{A})$.*

Proof. We can suppose that X is affine. Then the proof just repeats that of [5, Ch. VII, § 3, Theorem 3] with slight and obvious changes. □

Lemma 6.3. *Any non-commutative curve $(X, \mathcal{A})$ has enough invertible modules. Namely, the set*

$$\mathbf{L}_{\mathcal{A}} = \{\, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L} \mid \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module} \,\}$$

generates $\mathsf{Qcoh}\,\mathcal{A}$ (*hence, generates* $\mathscr{D}\mathcal{A}$).

Proof. We must show that if $\mathcal{M}' \subset \mathcal{M}$ is a proper submodule, there is a homomorphism $f : \mathcal{L} \to \mathcal{M}$ such that $\operatorname{Im} f \nsubseteq \mathcal{M}'$. As $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$, we can suppose that $\mathcal{A} = \mathcal{O}_X$. Moreover, as every $\mathcal{A}$-module is a direct limit of its coherent submodules, we can suppose that $\mathcal{M}$ is coherent. Let first $\mathcal{M}' \nsupseteq \operatorname{tors} \mathcal{M}$. Choose $x \in X_{\mathrm{cl}}$ such that $\operatorname{tors} \mathcal{M}_x \nsubseteq \mathcal{M}'_x$ and let $u_x \in \operatorname{tors} \mathcal{M}_x \setminus \mathcal{M}'_x$. There is a global section $u \in \Gamma(X, \operatorname{tors} \mathcal{M}) \subseteq \Gamma(X, \mathcal{M})$ such that u_x is its image in $\mathcal{M}_x$. Then there is a homomorphism $f : \mathcal{O}_X \to \mathcal{M}$ such that $f(1) = u$, so $\operatorname{Im} f \nsubseteq \mathcal{M}'$.

Let now $\mathcal{M}' \supseteq \operatorname{tors} \mathcal{M}$. Since $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{L}, \operatorname{tors} \mathcal{M}) = 0$ for any locally projective module $\mathcal{L}$, the map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M}/\operatorname{tors} \mathcal{M})$ is surjective. Hence, we can suppose that $\mathcal{M}$ is torsion free. Let $\mathcal{M}_y \neq \mathcal{M}'_y$ for some $y \in X_{\mathrm{cl}}$ and $u_y \in \mathcal{M}_y \setminus \mathcal{M}'_y$. There is a homomorphism $\varphi : \mathcal{K} \to \mathcal{KM}$ such that $\varphi(1) = u_y$. Let $\mathcal{N} = \varphi(\mathcal{O}_X)$. The set $S = \{\, x \in X_{\mathrm{cl}} \mid \mathcal{N}_x \nsubseteq \mathcal{M}_x \,\}$ is finite; moreover, $y \notin S$. For every $x \in S$ there is an ideal $L_x \subseteq \mathcal{O}_{X,x}$ such that $L_x \simeq \mathcal{O}_{X,x}$ and $\varphi(L_x) \subseteq \mathcal{M}_x$. Now choose an ideal $\mathcal{L} \subseteq \mathcal{O}_X$ such that $\mathcal{L}_x = L_x$ for $x \in S$ and $\mathcal{L}_x = \mathcal{O}_{X,x}$ otherwise. It is an invertible ideal, $\varphi(\mathcal{L}) \subseteq \mathcal{M}$ and $\varphi(\mathcal{L}) \nsubseteq \mathcal{M}'$. □

We will use the duality for left and right coherent torsion free $\mathcal{A}$-modules established in the following theorem.

Theorem 6.4. (1) *There is a* canonical $\mathcal{A}$-module, *i.e. such a module $\omega_{\mathcal{A}} \in \mathsf{tf}\,\mathcal{A}$ that* $\operatorname{inj.dim}_{\mathcal{A}} \omega_{\mathcal{A}} = 1$ *and $\mathcal{E}nd_{\mathcal{A}}\, \omega_{\mathcal{A}} \simeq \mathcal{A}^{\mathrm{op}}$ (so $\omega_{\mathcal{A}}$ can be considered as an* $\mathcal{A}$-bimodule). *Moreover, $\omega_{\mathcal{A}}$ is isomorphic as a bimodule to an ideal of $\mathcal{A}$.*

We denote by $\mathcal{M}^*$, where $\mathcal{M} \in \mathsf{Qcoh}\,\mathcal{A}$ (or $\mathcal{M} \in \mathsf{Qcoh}\,\mathcal{A}^{\mathrm{op}}$) the $\mathcal{A}^{\mathrm{op}}$-module (respectively, $\mathcal{A}$-module) $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}})$ (respectively, $\mathcal{H}om_{\mathcal{A}^{\mathrm{op}}}(\mathcal{M}, \omega_{\mathcal{A}})$).

(2) *The natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism for every $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ (or $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}^{\mathrm{op}}$) and the functors $\mathcal{M} \mapsto \mathcal{M}^*$ establish an exact duality of the categories $\mathsf{tf}\,\mathcal{A}$ and $\mathsf{tf}\,\mathcal{A}^{\mathrm{op}}$. Moreover, if $\mathcal{M} \in \mathsf{Coh}\,\mathcal{A}$, then $\mathcal{M}^{**} \simeq \mathcal{M}/\operatorname{tors} \mathcal{M}$.*

Proof. Each local ring $\mathcal{O}_x = \mathcal{O}_{X,x}$ is excellent, so its integral closure in $\mathcal{K}_x$ is finitely generated and its completion $\hat{\mathcal{O}}_x$ is reduced. Therefore $\mathcal{O}_x$ has a

canonical module ω_x which can be considered as an ideal in $\mathcal{O}_x$ [22, Korollar 2.12]. Moreover, $\mathcal{O}_x$ is normal for almost all $x \in X_{\mathrm{cl}}$ and in this case we can take $\omega_x = \mathcal{O}_{X,x}$. By Lemma 6.2, there is an ideal $\omega_X \subseteq \mathcal{O}_X$ such that $\omega_{X,x} = \omega_x$ for each $x \in X$. Then $\operatorname{inj.dim}_{\mathcal{O}_X} \omega_X = \sup \left\{ \operatorname{inj.dim}_{\mathcal{O}_{X,x}} \omega_x \right\} = 1$. As the natural map $\mathcal{O}_{X,x} \to \operatorname{End}_{\mathcal{O}_{X,x}} \omega_x$ is an isomorphism for each $x \in X$, the natural map $\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X} \omega_X$ is an isomorphism too. Therefore, ω_X is a canonical $\mathcal{O}_X$-module. Then it is known that the functor $\mathcal{M} \mapsto \mathcal{M}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$ is an exact self-duality of $\mathsf{tf}\, \mathcal{O}_X$ and the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism. Set now $\omega_{\mathcal{A}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$ for any $\mathcal{A}$-module $\mathcal{M}$, whence all statements of the theorem follow. □

As usually, we say that two non-commutative schemes $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are *Morita equivalent* if their categories of quasi-coherent modules are equivalent. A coherent locally projective $\mathcal{A}$-module $\mathcal{P}$ is said to be a *local progenerator* if $\mathcal{P}_x$ is a progenerator for $\mathcal{A}_x$ for all $x \in X$. It follows from Theorem 4.3 that then $(X, \mathcal{A})$ is Morita equivalent to $(X, \mathcal{E})$, where $\mathcal{E} = (\mathcal{E}nd_{\mathcal{A}} \mathcal{P})^{\mathrm{op}}$.

Theorem 6.5. (1) *Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are two non-commutative curves such that $\mathcal{A}_x$ is Morita equivalent to $\mathcal{B}_x$ for every $x \in X_{\mathrm{cl}}$. Then $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are Morita equivalent.*

(2) *Let now $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two central non-commutative curves finite over a field. If they are Morita equivalent, there is an isomorphism $\tau : X \xrightarrow{\sim} Y$ such that, for every points $x \in X$ and $y = \tau(x)$, the rings $(\tau^* \mathcal{B})_x$ and $\mathcal{A}_x$ are Morita equivalent.*

Proof. (1) If $\mathcal{A}_x$ and $\mathcal{B}_x$ are Morita equivalent, there is a progenerator P_x for $\mathcal{A}_x$ such that $\mathcal{B}_x \simeq (\operatorname{End}_{\mathcal{A}_x} P_x)^{\mathrm{op}}$. There is a $\mathcal{K}\mathcal{A}$-module $\mathcal{V}$ such that $\mathcal{V} \simeq \mathcal{K} P_x$ for all $x \in X_{\mathrm{cl}}$. Choose a normal over-ring $\mathcal{A}'$ of $\mathcal{A}$ and a coherent $\mathcal{A}'$-submodule $\mathcal{M} \subset \mathcal{V}$ such that $\mathcal{K}\mathcal{M} = \mathcal{V}$. Then $\mathcal{M}$ is a local progenerator for $\mathcal{A}'$. Set $\mathcal{B}' = (\mathcal{E}nd_{\mathcal{A}'} \mathcal{M})^{\mathrm{op}}$ and $S = \{ x \in X_{\mathrm{cl}} \mid \mathcal{A}_x \neq \mathcal{A}'_x \text{ or } \mathcal{B}_x \neq \mathcal{B}'_x \}$. The set S is finite, so there is an $\mathcal{A}$-submodule $\mathcal{P} \subset \mathcal{V}$ such that $\mathcal{P}_x = P_x$ for $x \in S$ and $\mathcal{P}_x = \mathcal{M}_x$ for $x \notin S$. Then $\mathcal{P}$ is a local progenerator for $\mathcal{A}$ and $\mathcal{B} \simeq (\mathcal{E}nd_{\mathcal{A}} \mathcal{P})^{\mathrm{op}}$.

(2) follows from [3, Section 6]. □

6.2. **Hereditary non-commutative curves.** We call a noetherian non-commutative scheme $(X, \mathcal{A})$ *hereditary* if all localizations $\mathcal{A}_x$ are hereditary rings, i.e. $\operatorname{gl.dim} \mathcal{A}_x = 1$. Then $\operatorname{gl.dim} \mathcal{A} = 1$ too, so $\operatorname{Ext}^2_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$ for all $\mathcal{A}$-modules $\mathcal{M}, \mathcal{N}$. Suppose that $(X, \mathcal{A})$ is a hereditary non-commutative curve. Then any torsion free coherent $\mathcal{A}$-module $\mathcal{M}$ is locally projective, so $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$ for any $\mathcal{A}$-module $\mathcal{N}$. If $\mathcal{N}$ is coherent and torsion, it implies that $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$. Therefore, every coherent $\mathcal{A}$-modules $\mathcal{M}$ splits as $\mathcal{M} = \operatorname{tors} \mathcal{M} \oplus \mathcal{M}'$, where $\mathcal{M}'$ is torsion free, hence locally projective. If a central non-commutative curve $(X, \mathcal{H})$ is hereditary, then X

is smooth. There is an effective description of hereditary non-commutative curves up to Morita equivalence.

First consider the case when $X = \operatorname{Spec} \boldsymbol{O}$, where $\boldsymbol{O}$ is a complete discrete valuation ring with the field of fractions $\boldsymbol{K}$, the maximal ideal $\mathfrak{m}$ and the residue field $\Bbbk = \boldsymbol{O}/\mathfrak{m}$. Let $\boldsymbol{H}$ be a hereditary reduced $\boldsymbol{O}$-algebra which is torsion free as $\boldsymbol{O}$-module. Then $\boldsymbol{K}\boldsymbol{H} \simeq \operatorname{Mat}(n, \boldsymbol{D})$, where $\boldsymbol{D}$ is a finite dimensional division algebra over $\boldsymbol{K}$. There is a unique maximal $\boldsymbol{O}$-order $\Delta \subset \boldsymbol{D}$ [30, Theorem 12.8]. It contains a unique maximal ideal $\mathfrak{M}$, which is both left and right principal ideal. Let $n = \sum_{i=1}^{k} n_i$ for some positive integers n_i, $\mathbf{n} = (n_1, n_2, \dots, n_k)$ and $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D})$ be the subring of $\operatorname{Mat}(n, \Delta)$ consisting of $k \times k$ block matrices (A_{ij}) such that A_{ij} is of size $n_i \times n_j$ and if $j > i$ all coefficients of A_{ij} are from $\mathfrak{M}$. Let also $L = \Delta^n$ considered as $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D})$-module and L_i be the submodule in L consisting of such vectors $(\alpha_1, \alpha_2, \dots, \alpha_n)$ that $\alpha_i \in \mathfrak{M}$ for $i > \sum_{j=1}^{i} n_i$. In particular, $L_0 = L$ and $L_k = \mathfrak{M}^n \simeq L$. If necessary, we denote $L_i = L_i(\boldsymbol{H})$.

Theorem 6.6 ([30, Theorem 39.14])**.** *Let $\boldsymbol{O}$ be a complete discrete valuation ring.*

(1) *Every connected hereditary $\boldsymbol{O}$-order is isomorphic to $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D})$ for some tuple $\mathbf{n} = (n_1, n_2, \dots, n_k)$, which is uniquely determined up to a cyclic permutation.*
(2) *Hereditary orders $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D})$ and $\boldsymbol{H}(\mathbf{n}', \boldsymbol{D}')$ are Morita equivalent if and only if $\boldsymbol{D} \simeq \boldsymbol{D}'$ and $\mathbf{n}$ and $\mathbf{n}'$ are of the same length.*
(3) *L_i $(0 \le i < k)$ are all indecomposable projective $\boldsymbol{H}$-modules and $U_i = L_i/L_{i+1}$ are all simple $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D})$-modules (up to isomorphism).*

Let now $(X, \mathcal{H})$ be a connected central hereditary non-commutative curve. Then $\mathcal{K}\mathcal{H}$ is a central simple $\mathcal{K}$-algebra: $\mathcal{K}\mathcal{A} = \operatorname{Mat}(n, \mathcal{D})$, where $\mathcal{D}$ is a central division algebra. For every closed point $x \in X$ the completion $\hat{\mathcal{D}}_x$ is isomorphic to $\operatorname{Mat}(m_x, \boldsymbol{D}_x)$ for some central division algebra $\boldsymbol{D}_x$ over $\hat{\mathcal{K}}_x$ and some integer $m_x = m_x(\mathcal{D})$. Therefore, for every closed point $x \in X$, $\hat{\mathcal{H}}_x$ is isomorphic to $\boldsymbol{H}(\mathbf{n}, \boldsymbol{D}_x)$ for some $\mathbf{n} = (n_1, n_2, \dots, n_k)$, where $\sum_{i=1}^{k} n_i = m_x n$. Thus Theorems 6.5 and 6.6 give the following result.

Theorem 6.7. *A central hereditary non-commutative curve $(X, \mathcal{H})$ is determined up to Morita equivalence by a central division $\mathcal{K}$-algebra $\mathcal{D}$ and a function $\kappa : X_{\mathrm{cl}} \to \mathbb{N}$ such that $\kappa(x) = 1$ for almost all $x \in X_{\mathrm{cl}}$.*

Remark 6.8. Representatives of a class given by $\mathcal{D}$ and κ can be obtained as follows. Choose an integer n such that $\kappa(x) \le n m_x(\mathcal{D})$ for all $x \in X_{\mathrm{cl}}$. There is an $\mathcal{O}_x$-order $\boldsymbol{H}_x$ in $\operatorname{Mat}(n, \mathcal{D})$ such that $\hat{\boldsymbol{H}}_x = \boldsymbol{H}(\mathbf{n}_x, \boldsymbol{D}_x)$ for some $\mathbf{n}_x = (n_{1,x}, n_{2,x}, \dots, n_{\kappa(x),x})$. Fix a normal non-commutative curve $(X, \boldsymbol{\Delta})$ such that $\mathcal{K}\boldsymbol{\Delta} = \mathcal{D}$. Then we can define $\mathcal{H} = \mathcal{H}(\mathbf{n}, \mathcal{D})$ as the non-commutative curve such that $\mathcal{K}\mathcal{H} = \operatorname{Mat}(n, \mathcal{D})$, $\mathcal{H}_x = \operatorname{Mat}(n, \boldsymbol{\Delta}_x)$ if $\kappa(x) = 1$ and $\mathcal{H}_x = \boldsymbol{H}_x$ if $\kappa(x) > 1$.

Let $S = \{ x \in X \mid \kappa(x) > 1 \}$, $\mathcal{L} = \boldsymbol{\Delta}^n$ considered as $\mathcal{H}$-module. Consider the submodules $\mathcal{L}_{x,i}$ $(0 \le i \le \kappa(x))$ such that $(\widehat{\mathcal{L}_{x,i}})_x = L_i(\hat{\boldsymbol{H}}_x)$ and $(\mathcal{L}_{x,i})_y = \mathcal{L}_y$ if $y \ne x$. Let also $U_{x,i} = \mathcal{L}_{x,i}/\mathcal{L}_{x,i+1}$ $(0 \le i < \kappa(x))$. Then $U_{x,i}$ are all simple $\mathcal{H}$-modules (up to isomorphism). Note that $\mathcal{L}_{x,0} = \mathcal{L}$ for every point x.

Theorem 6.9. *Let $\mathcal{H} = \mathcal{H}(\mathbf{n}, \mathcal{D})$.*

(1) *The set*

$$\mathbb{L}_{\mathcal{H}} = \{\mathcal{L}\} \cup \{ \mathcal{L}_{x,i} \mid x \in S,\, 1 \le k \le \kappa(x) \}$$

classically generates $\mathscr{D}^{\mathrm{c}}\mathcal{H}$, hence generates $\mathscr{D}\mathcal{H}$ (see [26, Theorem 2.2]*).*

(2) *$\mathscr{D}\mathcal{A} \simeq \mathscr{D}\mathbb{A}$, where $\mathbb{A}$ denotes the DG-category with the set of objects $\mathbb{L}_{\mathcal{A}}$ and $\mathbb{A}(\mathcal{L}', \mathcal{L}'') = \mathrm{RHom}_{\mathcal{A}}(\mathcal{L}', \mathcal{L}'')$.*

Proof. (1) Obviously, $\langle \mathbb{L}_{\mathcal{H}} \rangle_\infty$ contains all simple $\mathcal{H}$-modules. Therefore, it contains all torsion coherent $\mathcal{H}$-modules, as well as all coherent $\mathcal{H}$-submodules of $\mathcal{K}\mathcal{L}$. If $\mathcal{M}$ is a coherent torsion free $\mathcal{H}$-module, it contains a submodule $\mathcal{N}$ isomorphic to a submodule of $\mathcal{K}\mathcal{L}$ such that $\mathcal{M}/\mathcal{N}$ is also torsion free. It implies that $\langle \mathbb{L}_{\mathcal{H}} \rangle_\infty$ contains all coherent $\mathcal{H}$-modules, hence coincides with $\mathscr{D}^{\mathrm{c}}\mathcal{H}$.

(2) follows now from [26, Proposition 2.6]. □

Corollary 6.10. *Let $\Bbbk$ be an algebraically closed field.*

(1) *A connected hereditary algebraic non-commutative curve over $\Bbbk$ is defined up to Morita equivalence by a pair (X, κ), where X is a smooth connected algebraic curve over $\Bbbk$ and $\kappa : X_{\mathrm{cl}} \to \mathbb{N}$ is a function such that $\kappa(x) = 1$ for almost all x. Representatives of the Morita class given by such a pair are $\mathcal{H}(\mathbf{n}, \mathcal{K})$ as described in Remark 6.8.*

(2) *Two connected hereditary non-commutative curves given by the pairs (X, κ) and (X', κ') are Morita equivalent if and only if there is an isomorphism $\tau : X \to X'$ such that $\kappa'(\tau(x)) = \kappa(x)$ for all $x \in X_{\mathrm{cl}}$.*

In this case we write $\mathcal{H}(\mathbf{n}, X)$ instead of $\mathcal{H}(\mathbf{n}, \mathcal{K})$.

Proof. The Brauer group of $\mathcal{K}$ is trivial [25, Theorem 17]. Therefore, the algebra $\mathcal{D}$ in Theorem 6.7 coincides with $\mathcal{K}$. □

We say that a central non-commutative curve $(X, \mathcal{A})$ is *rational* (over a field $\Bbbk$) if all simple components of the algebra $\mathcal{K}\mathcal{A}$ are of the form $\mathrm{Mat}(n, \Bbbk(x))$. Then the curve X is also rational over $\Bbbk$.

Theorem 6.11. *Let $(X, \mathcal{H})$ be a connected rational hereditary non-commutative curve over a field $\Bbbk$ and $\kappa : X_{\mathrm{cl}} \to \mathbb{N}$ be the corresponding function. Let $S = \{ x \in X_{\mathrm{cl}} \mid \kappa(x) > 1 \}$, $o \in X_{\mathrm{cl}}$ be an arbitrary point.*

(1) *The set*

$$\overline{\mathbb{L}}_{\mathcal{H}} = \{ \mathcal{L}, \mathcal{L}(-o) \} \cup \{ \mathcal{L}_{x,i} \mid x \in S,\, 1 \le i < \kappa(x) \}$$

classically generates $\mathscr{D}^{\mathrm{c}}\mathcal{H}$, hence generates $\mathscr{D}\mathcal{H}$.

(2) *If $\mathcal{L}', \mathcal{L}'' \in \overline{\mathbb{L}}_{\mathcal{H}}$, then $\operatorname{Ext}^k_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = 0$ for all $k > 0$, while*

$$\dim \operatorname{Hom}_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = \begin{cases} 1 & \text{if } \mathcal{L}' = \mathcal{L}'', \\ & \text{or } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}_{x,i}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,i},\ \mathcal{L}'' = \mathcal{L}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}_{x,i},\ j > i, \\ 2 & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}, \\ 0 & \text{in all other cases.} \end{cases}$$

In particular, $\overline{\mathbb{L}}_{\mathcal{H}}$ is a tilting set for the category $\mathscr{D}\mathcal{H}$.

(3) *If $\theta_{x,i}$ are generators of the spaces $\operatorname{Hom}_{\mathcal{H}}(\mathcal{L}_{x,i}, \mathcal{L}_{x,i-1})$ $(1 \le i \le \kappa(x))$, then the products $\theta_x = \theta_{x,1}\theta_{x,2}\dots\theta_{x,\kappa(x)}$ are non-zero and any two of them generate $\operatorname{Hom}_{\mathcal{H}}(\mathcal{L}(-o), \mathcal{L})$.*

Proof. (1) If $X \simeq \mathbb{P}^1$, then all sheaves $\mathcal{O}(-x)$, hence all sheaves $\mathcal{L}(-x)$ are isomorphic. Moreover, in this case $\mathcal{L}_{x,\kappa(x)} \simeq \mathcal{L}(-x)$ for any $x \in X_{\mathrm{cl}}$, so we can apply Theorem 6.9.

(2) From the definition of $\mathcal{L}$ and $\mathcal{L}_{x,i}$ it immediately follows that

$$\mathcal{H}om_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') \simeq \begin{cases} \mathcal{O} & \text{if } \mathcal{L}' = \mathcal{L}'', \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,i},\ \mathcal{L}'' = \mathcal{L}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}_{x,i},\ j > i, \\ \mathcal{O}(o-x) & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}_{x,i}, \\ \mathcal{O}(o) & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}, \\ \mathcal{O}(-o) & \text{in all other cases.} \end{cases}$$

Since $\operatorname{Ext}^i_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = H^i(\mathcal{H}om_{\mathcal{H}}(\mathcal{L}', \mathcal{L}''))$, it implies the statement.

(3) One easily sees that, if $x = (1 : \xi)$ as the point of $\mathbb{P}^1$, then θ_x, up to a scalar, is the multiplication by $t - \xi$, where t is the affine coordinate on the affine chart $\mathbb{A}^1_0$. Now the statement is obvious. □

Recall that a *canonical algebra* by Ringel [31, 3.7] is given by a sequence of integers $(k_1, k_2, \dots, k_r)$, where $r \ge 2$ and all $k_i \ge 2$ if $r > 2$, and a sequence $(\lambda_3, \lambda_4, \dots, \lambda_r)$ of different non-zero elements from $\Bbbk$ (if $r = 2$, this sequence is empty). Namely, this algebra, which we denote by $\boldsymbol{R}(k_1, k_2, \dots, k_r; \lambda_3, \dots, \lambda_r)$, is given by the quiver

(6.2.1)
$$\begin{tikzcd}
& \bullet \arrow[r, "\alpha_{21}"] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{k_1-1,1}"] & \bullet \arrow[dr, "\alpha_{k_1 1}"] & \\
\bullet \arrow[ur, "\alpha_{11}"] \arrow[r, "\alpha_{12}"'] \arrow[dddr, "\alpha_{1r}"'] & \bullet \arrow[r, "\alpha_{22}"'] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{k_2-1,2}"'] & \bullet \arrow[r, "\alpha_{k_2 2}"'] & \bullet \\
& & & \vdots & & & \\
& & & & & & \\
& \bullet \arrow[r, "\alpha_{2r}"'] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{k_r-1,r}"'] & \bullet \arrow[uuur, "\alpha_{k_r r}"'] &
\end{tikzcd}$$

with relations $\alpha_j = \alpha_1 + \lambda_j\alpha_2$ for $3 \le j \le r$, where $\alpha_j = \alpha_{k_j j}\dots\alpha_{2j}\alpha_{1j}$. Certainly, if $r = 2$, it is the path algebra of a quiver of type $\tilde{\mathrm{A}}_{k_1+k_2}$. In particular, if $r = 2$, $k_1 = k_2 = 1$, it is the Kronecker algebra.

Corollary 6.12. *Let $(X,\mathcal{H})$ be a rational projective hereditary non-commutative curve, $\kappa: X_{\mathrm{cl}} \to \mathbb{N}$ be the corresponding function. Let $\mathcal{T} = \bigoplus_{\mathcal{F}\in\bar{\mathbb{L}}_{\mathcal{H}}} \mathcal{F}$ and $\Lambda = (\mathrm{End}_{\mathcal{H}}\mathcal{T})^{\mathrm{op}}$. If $S = \{x_1, x_2, \dots, x_r\}$ with $r \geq 2$, we set $k_i = \kappa(x_i)$. If $S = \{x\}$, we set $r = 2$, $k_2 = 1$ and $k_1 = \kappa(x)$. If $S = \emptyset$, we set $r = 2$, $k_1 = k_2 = 1$.*

(1) *$\Lambda \simeq \boldsymbol{R}(k_1, k_2, \dots, k_r; \lambda_3, \dots, \lambda_r)$ for some $\lambda_3, \dots, \lambda_r$.*
(2) *The functor $\mathrm{Hom}_{\mathcal{H}}(\mathcal{T}, \_)$ induces an equivalence $\mathscr{D}\mathcal{H} \simeq \mathscr{D}\Lambda$.*

Actually, the preceding considerations also show that a rational projective hereditary non-commutative curve is Morita equivalent to a *weighted projective line* by Geigle–Lenzing [17]. It can also be deduced from the description of hereditary non-commutative curves and the remark on page 271 of [17].

6.3. **Subhereditary non-commutative curves.**

Definition 6.13. A non-commutative curve $(X,\mathcal{A})$ is said to be *subhereditary* if there is a hereditary over-ring $\mathcal{H}$ of $\mathcal{A}$ and an ideal $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is semi-simple and $\mathcal{H}\mathcal{I} = \mathcal{I}$.

Obviously, we can suppose that $\mathcal{I}$ is the *conductor* of $\mathcal{H}$ in $\mathcal{A}$, i.e. $\mathcal{I} = \{ a \in \mathcal{K}\mathcal{A} \mid \mathcal{H}a \subseteq \mathcal{A} \} \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{H}, \mathcal{A})$. Note that the non-commutative curve $\mathcal{H}$ need not be connected. If $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_s$ are its connected components, we set $\mathbb{L}_{\mathcal{H}} = \bigcup_{i=1}^{s} \mathbb{L}_{\mathcal{H}_i}$.

Corollary 6.14. *Let $\mathcal{A}$ be a subhereditary non-commutative curve, $\mathcal{H}$ be its hereditary over-ring such that $\mathcal{Q} = \mathcal{A}/\mathcal{I}$ is semi-simple, where $\mathcal{I}$ is the conductor of $\mathcal{H}$ in $\mathcal{A}$. Let $\mathcal{A}_{\mathcal{H}} = \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \oplus \mathcal{H})$* (see Example 4.8).

(1) gl.dim $\mathcal{A}_{\mathcal{H}} \leq 2$.
(2) *There are semi-orthogonal decompositions $(\mathscr{T}, \mathscr{T}_1)$ and $(\mathscr{T}_1', \mathscr{T})$ of $\mathscr{D}\mathcal{A}_{\mathcal{H}}$ such that $\mathscr{T} = \mathscr{D}\mathcal{Q}$ and $\mathscr{T}_1 \simeq \mathscr{T}_1' \simeq \mathscr{D}\mathcal{H}$.*
Note that $\mathscr{T}_1 \neq \mathscr{T}_1'$.
(3) *The set $\{\mathcal{Q}\} \cup \mathbb{L}_{\mathcal{H}}$* (see Theorem 6.9) *generates $\mathscr{D}\mathcal{A}$.*

This corollary generalizes some results of [8].

Proof. Note that
$$\mathcal{A}_{\mathcal{H}} = \begin{pmatrix} \mathcal{A} & \mathcal{H} \\ \mathcal{I} & \mathcal{H} \end{pmatrix}.$$
Let $\mathcal{P} = \left(\begin{smallmatrix} \mathcal{H} \\ \mathcal{H} \end{smallmatrix}\right)$ considered as $\mathcal{A}_{\mathcal{H}}$-module. Then $\mathcal{E}nd_{\mathcal{A}_{\mathcal{H}}}\mathcal{P} \simeq \mathcal{H}$ and
$$\mathcal{I}_{\mathcal{P}} = \begin{pmatrix} \mathcal{I} & \mathcal{H} \\ \mathcal{I} & \mathcal{H} \end{pmatrix},$$
so $\mathcal{A}_{\mathcal{H}}/\mathcal{I}_{\mathcal{P}} \simeq \mathcal{Q}$ and we identify them. Then $(\mathcal{A}_{\mathcal{H}}, \mathcal{P}, \mathcal{Q})$ is a heredity relating chain berween $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{Q}$ (see Definition 4.10). Hence (1) and (2) follow from Corollary 4.11 and Corollary 4.12. Moreover, the set $\{\mathcal{Q}\} \cup \{ \mathcal{P} \otimes_{\mathcal{H}} \mathcal{L} \mid \mathcal{L} \in \mathbb{L}_{\mathcal{H}} \}$ generates $\mathscr{D}\mathcal{A}_{\mathcal{H}}$. As $\mathsf{G}_{\mathcal{H}}\mathcal{Q} = \mathcal{Q}$ and $\mathsf{G}_{\mathcal{H}}(\mathcal{P} \otimes_{\mathcal{H}} \mathcal{L}) \simeq \mathcal{L}$, we obtain the claim (3). $\square$

Corollary 6.15. *Suppose that the subhereditary non-commutative curve $\mathcal{A}$ is rational and keep the notations of Corollary 6.14. Let $\Lambda = (\operatorname{End}_{\mathcal{H}} \mathcal{T})^{\mathrm{op}}$, where $\mathcal{T} = \bigoplus_{\mathcal{F}\in\overline{\mathbb{L}}_{\mathcal{H}}} \mathcal{F}$, $\boldsymbol{Q} = \Gamma(X, \mathcal{Q})$, $\boldsymbol{E} = \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{Q}, \mathcal{T})$ and $\boldsymbol{A}$ be the algebra of triangular matrices*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{E} \\ 0 & \Lambda \end{pmatrix}.$$

Then $\{\mathcal{Q}[-1]\} \cup \mathsf{F}_{\mathcal{H}}\overline{\mathbb{L}}_{\mathcal{H}}$ is a tilting set in $\mathscr{D}^{\mathrm{c}}\mathcal{A}_{\mathcal{H}}$ and $\mathscr{D}\mathcal{A}_{\mathcal{H}} \simeq \mathscr{D}\boldsymbol{A}$. Therefore, the category $\mathscr{D}\mathcal{A}$ is a bilocalization of $\mathscr{D}\boldsymbol{A}$.

Note that the algebra Λ is isomorphic to a product of canonical algebras, so $\boldsymbol{A}$ is directed, hence quasi-hereditary.

References

[1] Alonso Tarrío L., Jeremías López A., Lipman J. Local homology and cohomology on schemes. Ann. Sci. Ecole Norm. Sup. 30 (1997) 1–39.

[2] Alonso Tarrío L., Jeremías López A., Souto Salorio M. J. Localization in categories of complexes and unbounded resolutions. Canadian Math. J. 52 (2000 225–247.

[3] Artin M, Zhang J. J. Noncommutative projective schemes. Adv. Math. 109 (1994) 228–287.

[4] Auslander M., Reiten I., Smalø S. Representation Theory of Artin Algebras. Cambridge Univ. Press, 1997.

[5] Bourbaki N. Algèbre commutatif. Chapitres 5 à 7. Hermann, Paris, 1975.

[6] Bourbaki N. Algèbre. Chapitre X. Algèbre homologique. Springer-Verlag, 1980.

[7] Bruns W., Herzog J. Cohen-Macaulay Rings. Cambridge University Press, 1993.

[8] Burban I., Drozd Y. Tilting on non-commutative rational projective curves. Math. Ann. 351 (2011) 665–709.

[9] Cartan H., Eilenberg S. Homological Algebra. Princeton University Press, 1956.

[10] Cline E., Parshall B., Scott L. Finite dimensional algebras and highest weight categories. J. reine angew. Math. 391 (1988) 85–99.

[11] Curtis C. W., Reiner I. Methods of Representation Theory. Vol. I. Wiley-Interscience, 1981.

[12] Dlab V., Ringel C. M. Quasi-hereditary algebras. Illinois J. Math., 33 (1989) 280–291.

[13] Drozd Y. A. Minors and theorems of reduction. Coll. Math. Soc. J. Bolyai, 6 (1973) 173–176.

[14] Drozd Y. A. On existence of maximal orders. Mat. Zametki, 37 (1985) 313–315.

[15] Drozd Y. A, Greuel G.-M. Tame and wild projective curves and classification of vector bundles. J. Algebra 246 (2001), no. 1, 1-54.

[16] Gabriel P. Des catégories abéliennes. Bull. Soc. Math. France, 90 (1962) 323–448.

[17] Geigle W., Lenzing H. A class of weighted projective curves arising in representation theory of finite dimensional algebras. Singularities, Representation of Algebras, and Vector Bundles. Lecture Notes Math. 1273, Springer-Verlag, 1987, 265–297.

[18] Geigle W., Lenzing H. Perpendicular categories with applications to representations and sheaves. J. Algebra 14 (1991) 273–343.

[19] Grothendieck A. Éléments de géométrie algébrique: I. Publ. Math. I.H.É.S. 4 (1960).

[20] Grothendieck A. Éléments de géométrie algébrique: II. Publ. Math. I.H.É.S. 8 (1961).

[21] Hartshorne R. Resudues and Duality. Lecture Notes in Math. 20. Springer-Verlag, 1966.

[22] Herzog J., Kunz E. (ed.) Der kanonische Modul eines Cohen-Macaulay-Rings. Lecture Notes Math. 238. Springer-Verlag, 1971.

[23] König S. Quasi-hereditary orders. Manus. Math. 68 (1990) 417–433.

[24] Kuznetsov A., Lunts V. A. Categorical resolutions of irrational singularities. arXiv:1212.6170 [math.AG].
[25] Lang S. On quasi-algebraic closure. Ann. Math. 55 (1952) 373–390.
[26] Lunts V. A. Categorical resolution of singularities. J. Algebra, 323 (2010) 2977–3003.
[27] Miyachi J. Localization of triangulated categories and derived categories. J. Algebra 141 (1991) 463–483
[28] Neeman A. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. J. Amer. Math. Soc. 9 (1996) 205–236
[29] Neeman A. Triangulated categories. Princeton University Press, 2001.
[30] Reiner I. Maximal Orders. Clarendon Press, 2003.
[31] Ringel C. M. Tame Algebras and Integral Quadratic Forms. Lecture Notes in Math. 1099, Springer-Verlag, 1984.
[32] Rouquier R. Dimensions of triangulated categories. J. K-Theory 1 (2008) 193–256.
[33] Spaltenstein N. Resolutions of unbounded complexes. Compositio Math. 65 (1988) 121-154.
[34] Van den Bergh M. Non-commutative crepant resolutions. The legacy of Niels Henrik Abel, 749770, Springer, 2004.

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